Time-varying linear systems: relative degree and normal form

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Abstract

We define the relative degree of time-varying linear systems, show that it coincides with Isidori's and with Liberzon/Morse/Sontag's definition if the system is understood as a time-invariant nonlinear system, characterize it in terms of the system data and their derivatives, derive a normal form with respect to a time-varying linear coordinate transformation, and finally characterize the zero dynamics.

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1 Introduction

The concept of relative degree goes back to single-input single-output linear systems described in the frequency domain by a transfer function $p(s)/q(s)$ where the relative degree is defined by $r = \deg q - \deg p$; p, q denote polynomials with real coefficients. To derive a characterization in the time domain, take any realization of $p(s)/q(s)$, say

$$\left.\begin{array}{rcl} \dot{x}(t) &=& A\,x(t) + b\,u(t) \\ y(t) &=& c\,x(t) \end{array}\right\} \tag{1.1}$$

with $A \in \mathbb{R}^{n\times n}$, $b, c^T \in \mathbb{R}^n$. Then $p(s)/q(s) = c(sI-A)^{-1}b = \sum_{k=0}^{\infty} cA^k b\, s^{-(k+1)}$, and it is easy to see that $r = \deg q - \deg p$ if, and only if,

$$\forall\, k = 0,\dots,r-2 \;:\; cA^k b = 0, \quad cA^{r-1}b \neq 0\,. \tag{1.2}$$

Isidori [3, p. 137] generalized the concept of relative degree to single-input single-output time-invariant nonlinear systems, affine in the control, of the form

$$\left.\begin{array}{rcl} \dot{x} &=& f(x) + g(x)\,u(t) \\ y(t) &=& h(x(t)), \end{array}\right\} \tag{1.3}$$

with $f, g \in \mathcal{C}^\ell(\mathbb{R}^n,\mathbb{R}^n)$, $h \in \mathcal{C}^\ell(\mathbb{R}^n,\mathbb{R})$, and $\ell \in \mathbb{N}$: the system (1.3) has relative degree $r \in \{1,\dots,\ell\}$ at $x^0 \in \mathbb{R}^n$ if, and only if, there exists an open neighbourhood $\mathcal{U}$ of x^0, such that,

$$\forall\, x \in \mathcal{U} \quad \forall\, k \in \{0,\dots,r-2\} \;:\; L_g L_f^k h(x) = 0, \quad L_g L_f^{r-1} h(x^0) \neq 0\,, \tag{1.4}$$

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where $L_f\lambda = \frac{\partial\lambda}{\partial x} f$ denotes the derivative of a function λ along a vector field f; see, for example, [3, Sect. 1.2].

The importance of the relative degree is that it leads to a normal form [3, Sec. 4.1]: if (1.3) has relative degree r at x^0, then there exists a diffeomorphism Φ, defined in a neighbourhood of x^0, which transforms (1.3) under $(\xi,\eta) = \Phi(x)$, $\xi = (y,\ldots,y^{(r-1)})^T$, to

$$\left.\begin{array}{rcl} y^{(r)} &=& L_f^r h(\Phi^{-1}(\xi,\eta)) + L_g L_f^{(r-1)} h(\Phi^{-1}(\xi,\eta))\, u(t) \\ \dot\eta &=& q(\xi,\eta) \\ y(t) &=& \xi_1(t)\,, \end{array}\right\} \tag{1.5}$$

for some $q \in \mathcal{C}^\ell(\mathbb{R}^n,\mathbb{R}^{n-r})$. This form gives immediately that, for $x(t_0) = x^0$, "the relative degree r is exactly equal to the number of times one has to differentiate the output $y(t)$ at time $t = t_0$ in order to have the value $u(t_0)$ of the input explicitly appearing"[3, p. 139]. Moreover, u enters only in a single differential equation in (1.5) directly and it is possible to read off the zero dynamics, see [3, Sec. 4.3] and Section 3 of the present paper.

The purpose of the present note is to generalize the concept of relative degree to time-varying linear systems of the form

$$\left.\begin{array}{rcl} \dot x &=& A(t)\, x + B(t)\, u(t) \\ y(t) &=& C(t)\, x(t), \end{array}\right\} \tag{1.6}$$

with $A \in \mathcal{C}^\ell(\mathbb{R},\mathbb{R}^{n\times n})$, $B, C^T \in \mathcal{C}^\ell(\mathbb{R},\mathbb{R}^{n\times m})$, $\ell \in \mathbb{N}$, and to derive a time-varying linear coordinate transformation which takes (1.6) to a normal form. To this end, we generalize the concept of relative degree to time-varying nonlinear systems first.

The paper is organized as follows. In Section 2, we present a definition of relative degree for time-varying nonlinear systems. It is shown that this definition coincides, if the system is time-invariant with Isidori's definition ([3, p. 220] respectively with the definition by Liberzon et al. [4, Def. 2]; furthermore, if the system is linear time-varying and viewed as a time-invariant nonlinear system, the definition coincides again with Isidori's definition. Our main result is a normal form for time-varying linear systems given in Section 3. In Section 4, we parameterize the zero dynamics of time-varying linear systems and character their stability properties. We have relegated a refined version of Doležal's Theorem to the Appendix, which is used in the antecedent proofs.

We close this introduction with remarks on notation. Throughout, $\mathbb{R}_{\geq 0} := [0,\infty)$ and $\|\cdot\|$ is the Euclidean inner product or the induced norm on $\mathbb{R}^n$; if $M_1, M_2 \in \mathbb{R}^{n\times n}$ are symmetric, then the notion $M_1 \geq M_2$ means $x^T M_1 x \geq x^T M_2 x$ for all $x \in \mathbb{R}^n$; $\mathrm{Gl}_n(\mathbb{R})$ denotes the general linear group of invertible matrices $A \in \mathbb{R}^{n\times n}$; $C^\ell(U,W)$ is the vector space of ℓ-times differentiable functions $f: U \to W$, U and W are open sets; and $\mathcal{L}^\infty(U,W)$ the set of essentially bounded functions $f: U \to W$.

2 Relative degree: definition and characterizations

For time-invariant nonlinear multi-input multi-output systems, affine in the control, of the form

$$\left.\begin{array}{rcl} \dot x &=& f(x) + g(x)\, u(t) \\ y(t) &=& h(x(t)), \end{array}\right\} \tag{2.1}$$

with $f \in \mathcal{C}^\ell(\mathbb{R}^n,\mathbb{R}^n)$, $g \in \mathcal{C}^\ell(\mathbb{R}^n,\mathbb{R}^{n\times m})$, $h \in \mathcal{C}^\ell(\mathbb{R}^n,\mathbb{R}^m)$, $\ell \in \mathbb{N}$, the strict relative degree is defined as follows.

Definition 2.1 (Isidori ([3, p. 220]))
Let $\mathcal{U} \subset \mathbb{R}^n$ be open and $r \in \{1, \ldots, \ell\}$. The time-invariant nonlinear system (2.1) has (*strict*) *relative degree* r *on* $\mathcal{U}$ if, and only if,

(i) $\forall\, \xi \in \mathcal{U}\ \forall\, k \in \{0, \ldots, r-2\}\ :\ L_g L_f^k h(\xi) = 0_{m\times m}$,

(ii) $\forall\, \xi \in \mathcal{U}\ :\ L_g L_f^{r-1} h(\xi) \in \mathrm{Gl}_m(\mathbb{R})$.

This definition is due to Isidori ([3, p. 220]) who defines it more general for a vector relative degree; we consider only the "strict" relative degree, that is (ii). The original definition in [3, p. 220] defines the relative degree at a point $\xi^0 \in \mathbb{R}^n$ and some neighbourhood; however, this is equivalent to Definition 2.1, the latter is technically easier to deal with in the following. In the single-input single-output case, the notion of "strict" is redundant and Isidori showed that "the relative degree r is exactly equal to the number of times one has to differentiate the output $y(t)$ at time $t = t_0$ in order to have the value $u(t_0)$ of the input explicitly appearing"[3, p. 139]. This latter characterization was formalized by Liberzon et al. [4] and related to output-input stability. We extend their notion of functions H_k to time-varying nonlinear systems of the form

$$\left.\begin{array}{rcl} \dot{x} &=& F(t, x, u(t)) \\ y(t) &=& H(t, x(t)) \end{array}\right\} \tag{2.2}$$

where

$$F \in \mathcal{C}^\ell(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m, \mathbb{R}^n)\,, \qquad H \in \mathcal{C}^\ell(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^m), \qquad \ell \in \mathbb{N}\,,$$

and define recursively, for $k = 0, 1, 2, \ldots, \ell-1$, the functions $H_0(t,x) := H(t,x)$,

$$\left.\begin{array}{rcrcl} H_{k+1} &:& \mathbb{R}\times\mathbb{R}^n\times(\mathbb{R}^m)^{k+1} &\to& \mathbb{R}^m \\ && (t, x, u_0, \ldots, u_k) &\mapsto& \frac{\partial H_k}{\partial t} + \frac{\partial H_k}{\partial x}\, F(t,x,u_0) + \sum\limits_{j=0}^{k-1} \frac{\partial H_k}{\partial u_j}\, u_{j+1}. \end{array}\right\} \tag{2.3}$$

This allows to express the k-th derivative of $y(t)$ in terms of t, $x(t)$ and $u(t), \ldots, u^{(k-1)}(t)$:

$$\forall\, t \in \mathbb{R} \quad \forall\, k \in \{1, \ldots, \ell-1\}\ :\ y^{(k)}(t) = H_k(t, x(t), u(t), \ldots, u^{(k-1)}(t))\,.$$

Definition 2.2 (Generalization of Liberzon et al. [4, Def. 2])
Let $\mathcal{T} \subset \mathbb{R}$ and $\mathcal{U} \subset \mathbb{R}^n$ be open sets, and $r \in \mathbb{N}$. Then a system (2.2) is said to have (*strict and uniform*) *relative degree* $r \in \mathbb{N}$ *on* $\mathcal{T}\times\mathcal{U}$ if, and only if,

(i) $\forall\, (t,x) \in \mathcal{T}\times\mathcal{U} \quad \forall\, k \in \{1, \ldots, r-1\}\ \forall\, i \in \{0, \ldots, k-1\}\ :\ \frac{\partial H_k}{\partial u_i}(t, x, u_0, \ldots, u_{k-1}) = 0_{m\times m}$;

(ii) $\forall\, (t, x, u_0) \in \mathcal{T}\times\mathcal{U}\times\mathbb{R}^m\ :\ \frac{\partial H_r}{\partial u_0}(t, x, u_0, \ldots, u_{r-1}) \in \mathrm{Gl}_m(\mathbb{R})$.

The notion 'strict' refers to the multivariable case where we do not allow for a relative degree vector $(r_1, \ldots, r_m) \in \mathbb{N}^m$ with different entries, see [3, Sect. 5.1], but assume that the matrices $\frac{\partial}{\partial u_i} H_k$ are either 'strictly' zero or invertible, globally on their domain.

Proposition 2.3 Let $\mathcal{U} \subset \mathbb{R}^n$ be an open set, $r, \ell \in \mathbb{N}$ with $r \le \ell$, and consider the time-invariant nonlinear system (2.1), affine in the control. Then (2.1) has relative degree r on $\mathcal{U}$ in the sense of Definition 2.1 if, and only if, (2.1) has relative degree r on $\mathbb{R}\times\mathcal{U}$ in the sense of Definition 2.2.

Proof: The following analysis is considered for $(t,x) \in \mathbb{R} \times \mathcal{U}$ only; for notational convenience, we omit to repeat this and also suppress the arguments of the functions for simplicity.
"$\Rightarrow$": Suppose that (2.1) has relative degree r in the sense of Definition 2.1. We first show by induction on k that the following holds:

$$\forall\, k \in \{1,\ldots,r-1\} \quad \forall\, i \in \{1,\ldots,k-1\} \;:\quad H_k = L_f^k h\,, \quad \frac{\partial H_k}{\partial u_i} = 0_{m\times m}\,. \tag{2.4}$$

We have, in view of (i) in Definition 2.1,

$$H_1 = \frac{\partial H_0}{\partial x}\,(f + g\,u_0) + \frac{\partial H_0}{\partial u_0}\,u_1 = L_f h + L_g h\, u_0 = L_f h\,,$$

and so $\frac{\partial H_1}{\partial u_0} = \frac{\partial}{\partial u_0} L_f h = 0_{m\times m}$. If (2.4) holds for all $k \le r-2$, then, in view of (i) in Definition 2.1,

$$H_{k+1} = \frac{\partial H_k}{\partial x}\,(f + g\,u_0) + \sum_{j=0}^{k-1} \frac{\partial H_k}{\partial u_j}\,u_{j+1} = L_f^{k+1}h + L_g L_f^k h\, u_0 = L_f^{k+1} h\,,$$

and so, for all $i \in \{1,\ldots,k\}$, $\frac{\partial H_{k+1}}{\partial u_i} = \frac{\partial}{\partial u_i} L_f^{k+1} h = 0_{m\times m}$. This proves (2.4), and therefore (i) of Definition 2.2 holds.
To prove (ii) in Definition 2.2, note that, in view of (2.4),

$$H_r = \frac{\partial H_{r-1}}{\partial x}\,(f + g\,u_0) + \sum_{j=0}^{r-2} \frac{\partial H_{r-1}}{\partial u_j}\,u_{j+1} = L_f^r h + L_g L_f^{r-1} h\, u_0\,,$$

and thus, invoking (ii) in Definition 2.1, $\frac{\partial H_r}{\partial u_0} = L_g L_f^{r-1} h \in \mathrm{Gl}_m(\mathbb{R})$. This proves (ii) in Definition 2.2.
"$\Leftarrow$": Suppose that (2.1) has relative degree r in the sense of Definition 2.2. We show first by induction on k that the following holds:

$$\forall\, k \in \{0,1,\ldots,r-2\} \;:\quad L_g L_f^k h = 0_{m\times m}\,, \quad H_{k+1} = L_f^{k+1} h\,. \tag{2.5}$$

We have

$$H_1 = \frac{\partial H_0}{\partial x}\,(f + g\,u_0) + \frac{\partial H_0}{\partial u_0}\,u_1 = \frac{\partial h}{\partial x}\,f + \frac{\partial h}{\partial x}\,g\,u_0 = L_f h + L_g h\,u_0\,,$$

and so, in view of (i) in Definition 2.2, $0_{m\times m} = \frac{\partial H_1}{\partial u_0} = L_g h$, which gives $H_1 = L_f h$. If (2.5) holds for all $k \le r-3$, then, again in view of (i) in Definition 2.2,

$$H_{k+2} = \frac{\partial H_{k+1}}{\partial x}\,(f + g\,u_0) + \sum_{j=0}^{k} \frac{\partial H_{k+1}}{\partial u_j} u_{j+1} = \frac{\partial}{\partial x} L_f^{k+1} h\, f + \frac{\partial}{\partial x} L_f^{k+1} h g\, u_0 = L_f^{k+2} h + L_g L_f^{k+1} h\, u_0\,,$$

and furthermore $0_{m\times m} = \frac{\partial H_{k+2}}{\partial u_0} = L_g L_f^{k+1} h$, which yields $H_{k+2} = L_f^{k+2} h$. This proves (2.5).
Finally, by (2.5), (i) in Definition 2.1 follows. Applying (2.5) again gives

$$H_r = \frac{\partial H_{r-1}}{\partial x}\,(f + g\,u_0) + \sum_{j=0}^{r-2} \frac{\partial H_{r-1}}{\partial u_j}\,u_{j+1}. = L_f^r h + L_g L_f^{r-1} h\, u_0\,,$$

and thus, invoking (ii) in Definition 2.2, $\frac{\partial H_r}{\partial u_0} = L_g L_f^{r-1} h \in \mathrm{Gl}_m(\mathbb{R})$. This proves (ii) in Definition 2.1 and completes the proof of the proposition. □

Remark 2.4

(i) It follows from the proof of Proposition 2.3 that the relative degree of the time-invariant system (2.1) does not depend on t: if its relative degree is defined on $\mathcal{T} \times \mathcal{U}$ for some open set $\mathcal{T} \subset \mathbb{R}$, then it is defined on $\mathbb{R} \times \mathcal{U}$.

(ii) It also follows from Proposition 2.3 and [4, Prop. 2] that for single-input single-output time-invariant systems (2.1), Definition 2.2 coincides with the definition of the relative degree given by Liberzon et al. [4, Def. 2].

The characterization of the relative degree for time-varying linear systems (1.6) in terms of the matrix-functions A, B, C and their derivatives relies crucially on the following right operator, a notational convention, which allows for a neat presentation.

Definition 2.5 For $\ell \in \mathbb{N}$, $A \in \mathcal{C}^\ell(\mathbb{R}, \mathbb{R}^{n\times n})$, and $C \in \mathcal{C}^\ell(\mathbb{R}, \mathbb{R}^{m\times n})$ set

$$\forall\, t \in \mathbb{R} \ : \ \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A(t)\right)_R \big(C(t)\big) := \dot{C}(t) + C(t)A(t)\,,$$

$$\forall\, t \in \mathbb{R} \quad \forall\, k \in \{1, \ldots, \ell\} \ : \ \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A(t)\right)_R^k \big(C(t)\big) := \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A(t)\right)_R \Big(\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A(t)\right)_R^{k-1} \big(C(t)\big)\Big)\,.$$

The sub-script R indicates that A acts on C by multiplication from the right.

Theorem 2.6 Let $r, \ell \in \mathbb{N}$ with $r \le \ell$, and $\mathcal{T} \subset \mathbb{R}$ be an open set. Then the time-varying linear system (1.6) has relative degree r on $\mathcal{T} \times \mathbb{R}^n$ if, and only if, (A, B, C) satisfy

$$\left.\begin{array}{r} \forall t \in \mathcal{T} \quad \forall\, k = 0, \ldots, r-2 \ : \ \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A(t)\right)_R^k \big(C(t)\big)\, B(t) = 0_{m\times m} \\ \forall\, t \in \mathcal{T} \ : \ \Big(\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A(t)\right)_R^{r-1} \big(C(t)\big) B(t)\Big) \in \mathrm{Gl}_m(\mathbb{R})\,. \end{array}\right\} \tag{2.6}$$

The proof of Theorem 2.6 depends on the following technicality.

Lemma 2.7 Let $r, \ell \in \mathbb{N}$ with $r \le \ell$. Then the functions H_k defined in (2.3) and applied to the linear time-varying system (1.6) satisfy, for all $k \in \{0, \ldots, \ell - 1\}$ and all $(t, x, u_0, \ldots, u_k) \in \mathbb{R} \times \mathbb{R}^n \times (\mathbb{R}^m)^{k+1}$,

$$\begin{aligned} &H_{k+1}(t, x, u_0, \ldots, u_k) \\ &\quad = \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A(t)\right)_R^{k+1} \big(C(t)\big)\, x \ + \ \sum_{j=0}^{k} \sum_{i=j}^{k} \tbinom{i}{j} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{i-j} \Big[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A(t)\right)_R^{k-i} \big(C(t)\big)\, B(t)\Big]\, u_j\,. \end{aligned} \tag{2.7}$$

Proof: Applying the definition of H_k to

$$F(t, x, u) := A(t)\, x + B(t)\, u \quad \text{and} \quad H(t, x) := C(t)\, x \qquad \text{for} \quad (t, x, u) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m,$$

we have

$$\begin{aligned} H_1(t, x, u_0) \ &= \ \tfrac{\partial}{\partial t} C(t)x \ + \ \tfrac{\partial}{\partial x}(C(t)x)\, [A(t)x + B(t)\, u_0] \\ &= \ \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A(t)\right)_R \big(C(t)\big)x \ + \ C(t)B(t)\, u_0 \\ &= \ \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A(t)\right)_R^1 \big(C(t)\big)x \ + \ \sum_{j=0}^{0} \sum_{i=j}^{0} \tbinom{i}{j} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{i-j} \Big[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A(t)\right)_R^{0-i} \big(C(t)\big) B(t)\Big]\, u_j, \end{aligned}$$

which shows (2.7) for $k = 0$. To prove (2.7) by induction over $k \in \{0, \dots, \ell - 1\}$, assume that (2.7) holds for all $k \in \{0, \dots, \ell - 2\}$. Then, suppressing the arguments of H_k and A, B, C for simplicity, we calculate

$$
\begin{aligned}
H_{k+1} &= \tfrac{\partial}{\partial t} H_k + \tfrac{\partial}{\partial x} H_k \, [Ax + B\, u_0] + \sum_{l=0}^{k-1} \tfrac{\partial}{\partial u_l} H_k \, u_{l+1} \\
&= \tfrac{\partial}{\partial t} \Bigg[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^k (C)\, x + \sum_{j=0}^{k-1} \sum_{i=j}^{k-1} \tbinom{i}{j} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{i-j} \Big[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k-1-i} (C)\, B \Big] \, u_j \Bigg] \\
&\quad + \tfrac{\partial}{\partial x} \Big[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^k (C)\, x \Big] [Ax + B\, u_0] \\
&\quad + \sum_{l=0}^{k-1} \tfrac{\partial}{\partial u_l} \Bigg[\sum_{j=0}^{k-1} \sum_{i=j}^{k-1} \tbinom{i}{j} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{i-j} \Big[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k-1-i} (C)\, B \Big] \, u_j \Bigg] u_{l+1} \\
&= \tfrac{\mathrm{d}}{\mathrm{dt}} \Big(\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^k (C) \Big)\, x + \sum_{j=0}^{k-1} \sum_{i=j}^{k-1} \tbinom{i}{j} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{i-j+1} \Big[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k-1-i} (C)\, B \Big] \, u_j \\
&\quad + \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^k (C)\, A\, x + \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^k (C)\, B\, u_0 \\
&\quad + \sum_{l=0}^{k-1} \sum_{i=l}^{k-1} \tbinom{i}{l} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{i-l} \Big[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k-1-i} (C)\, B \Big] \, u_{l+1} \\
&= \Big[\tfrac{\mathrm{d}}{\mathrm{dt}} \Big(\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^k (C) \Big) + \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^k (C)\, A \Big]\, x + \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^k (C)\, B\, u_0 \\
&\quad + \sum_{j=0}^{k-1} \sum_{i=j}^{k-1} \tbinom{i}{j} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{i-j+1} \Big[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k-1-i} (C)\, B \Big] \, u_j \\
&\quad + \sum_{j=1}^{k} \sum_{i=j-1}^{k-1} \tbinom{i}{j-1} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{i-j+1} \Big[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k-1-i} (C)\, B \Big] \, u_j \\
&= \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k+1} (C)\, x + \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^k (C)\, B\, u_0 \\
&\quad + \sum_{i=0}^{k-1} \tbinom{i}{0} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{i+1} \Big[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k-1-i} (C)\, B \Big] \, u_0 \\
&\quad + \sum_{j=1}^{k-1} \Bigg(\sum_{i=j}^{k-1} \Big[\tbinom{i}{j} + \tbinom{i}{j-1} \Big] \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{i-j+1} \Big[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k-1-i} (C)\, B \Big] \, u_j \\
&\qquad\qquad + \tbinom{j-1}{j-1} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{j-1-j+1} \Big[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k-1-j+1} (C)\, B \Big] \, u_j \Bigg) \\
&\quad + \sum_{i=k-1}^{k-1} \tbinom{i}{k-1} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{i-k+1} \Big[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k-1-i} (C)\, B \Big] \, u_k \\
&= \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k+1} (C)\, x + \underbrace{\sum_{i=-1}^{k-1} \tbinom{i}{0} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{i+1} \Big[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k-1-i} (C)\, B \Big] \, u_0}_{= \sum_{i=0}^{k} \binom{i}{0} \left(\frac{\mathrm{d}}{\mathrm{dt}}\right)^{i-0} \left[\left(\frac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k-i} (C)\, B \right] u_0} \\
&\quad + \sum_{j=1}^{k-1} \Bigg(\sum_{i=j}^{k-1} \tbinom{i+1}{j} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{i+1-j} \Big[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k-(i+1)} (C)\, B \Big] \, u_j \\
&\qquad\qquad + \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k-j} (C)\, B\, u_j \Bigg) \\
&\quad + \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{0} (C)\, B\, u_k
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k+1}(C)\,x + \sum_{i=0}^{k} \tbinom{i}{0} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{i-0} \left[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k-i}(C)\,B\right] u_0 \\
&\quad + \sum_{j=1}^{k-1} \Bigg(\sum_{i=j+1}^{k} \tbinom{i}{j} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{i-j} \left[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k-i}(C)\,B\right] u_j \\
&\qquad\qquad + \tbinom{j}{j} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{j-j} \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k-j}(C)\,B\,u_j \Bigg) \\
&\quad + \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{0}(C)\,B\,u_k \\
&= \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k+1}(C)\,x + \sum_{i=0}^{k} \tbinom{i}{0} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{i-0} \left[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k-i}(C)\,B\right] u_0 \\
&\quad + \sum_{j=1}^{k-1} \sum_{i=j}^{k} \tbinom{i}{j} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{i-j} \left[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k-i}(C)\,B\right] u_j \\
&\quad + \tbinom{k}{k} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{k-k} \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k-k}(C)\,B\,u_k \\
&= \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k+1}(C)\,x + \sum_{j=0}^{k} \sum_{i=j}^{k} \tbinom{i}{j} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{i-j} \left[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k-i}(C)\,B\right] u_j.
\end{aligned}
$$

This shows (2.7) for $k+1$ and therefore the proof of the lemma is complete. □

Proof of Theorem 2.6:
"⇒": Suppose that Definition 2.2 holds. We show the first condition in (2.6) by induction over $k \in \{1,\ldots,r-2\}$ (omitting the arguments of the operators). For $N = k = 0$ we have, by (2.7),

$$H_1 = \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R (C)\,x + CB\,u_0,$$

and so, by Definition 2.2, $0_{m\times m} = \frac{\partial H_1}{\partial u_0} = CB$. Suppose that the first condition in (2.6) holds for all $k = 0,\ldots,N$, where $N \le r-3$. Then (2.7) yields,

$$H_{N+2} = \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{N+2}(C)\,x + \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{N+1}(C)\,B\,u_0\,,$$

and so, invoking (i) of Definition 2.2,

$$0_{m\times m} = \frac{\partial H_{N+2}}{\partial u_0} = \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{N+1}(C)\,B\,.$$

This proves the first condition in (2.6).
To see the second condition in (2.6), note that (2.7) yields

$$H_r = \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{r}(C)\,x + \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{r-1}(C)\,B\,u_0\,,$$

and so, by (ii) of Definition 2.2,

$$\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{r-1}(C)\,B \;=\; \frac{\partial H_r}{\partial u_0} \;\in\; \mathrm{Gl}_m(\mathbb{R}),.$$

This completes the proof of (2.6).
"⇐": Suppose that (2.6) holds. Then (2.7) yields

$$\forall\, k = 0,\ldots,r-2 \;:\; H_{k+1} = \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{k+1}(C)\,x\,,$$

and thus (i) in Definition 2.2 follows. Finally, the second statement in (2.6) together with (2.7) gives

$$\frac{\partial H_r}{\partial u_0} = \frac{\partial}{\partial u_0}\Big(\big(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\big)_R^r (C)\, x + \big(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\big)_R^{r-1} (C)\, B\, u_0 \Big) = \big(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\big)_R^{r-1} (C)\, B \in \mathrm{Gl}_m(\mathbb{R})\,.$$

This proves (ii) of Definition 2.2 and completes the proof of the theorem. □

Remark 2.8

(i) If follows from the proof of Theorem 2.6 that the relative degree of the time-varying linear system (1.6) does not depend on x: if its relative degree is defined on some $\mathcal{T} \times U$, where $U \subset \mathbb{R}^n$ is open, then it is defined on $\mathcal{T} \times \mathbb{R}^n$. We therefore omit, at most places in the following, the second component in $\mathcal{T} \times \mathbb{R}^n$.

(ii) If A, B, C are real analytic matrices and the linear system (1.6) has relative degree r on $\mathcal{T} \times \mathbb{R}^n$ for some open $\mathcal{T} \subset \mathbb{R}$, then the Identity Theorem for analytic functions implies that (1.6) has relative degree r on $(\mathbb{R} \setminus D) \times \mathbb{R}^n$, where D denotes a discrete set.

(iii) If the linear system (1.6) is time-invariant, then Theorem 2.6 yields that (1.6) has relative degree $r \in \mathbb{N}$ on $\mathbb{R} \times \mathbb{R}^n$ if, and only if,

$$CA^kB = 0_{m\times m} \text{ for all } k = 0, \ldots, r-2 \quad \text{and} \quad CA^{r-1}B \in \mathrm{Gl}_m(\mathbb{R})\,.$$

This is the well known characterization of strict relative degree, see [3, Rem. 4.1.2] for single-input single-output systems.

Instead of defining the relative degree of the linear time-varying system (1.6) as in Definition 2.2, we may consider the equivalent description of (1.6) as a time-invariant nonlinear system and determine the relative degree according to Definition 2.1. In the following we will show that both definitions coincide. Introducing an additional variable z with initial condition $z(0) = 0$, (1.6) is equivalent to

$$\left.\begin{array}{rcl} \frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} x \\ z \end{pmatrix} & = & \begin{pmatrix} A(z)\, x \\ 1 \end{pmatrix} + \begin{pmatrix} B(z) \\ 0 \end{pmatrix} u(t) \\ y(t) & = & C(z(t))\, x(t). \end{array}\right\} \tag{2.8}$$

Proposition 2.9 Let $\mathcal{T} \subset \mathbb{R}$ and $\mathcal{U} \subset \mathbb{R}^n$ be open sets, $r, \ell \in \mathbb{N}$ with $r \leq \ell$. The time-varying linear system (1.6) has relative degree r on $\mathcal{T} \times \mathcal{U}$ in the sense of Definition 2.2 if, and only if, the equivalent nonlinear time-invariant system (2.8) has relative degree r on $\mathcal{U} \times \mathcal{T}$ in the sense of Definition 2.1.

Proof: Writing

$$f(x,z) = \begin{pmatrix} A(z)x \\ 1 \end{pmatrix}, \quad g(x,z) = \begin{pmatrix} B(z) \\ 0 \end{pmatrix}, \quad h(x,z) = C(z)\, x,$$

we show by induction over $N \in \{0, \ldots, \ell\}$ that

$$\forall\, (x,z) \in \mathcal{U} \times \mathcal{T}\ \forall\, k \in \{0, \ldots, \ell\}\ :\ L_f^k h(x,z) = \big(\tfrac{\mathrm{d}}{\mathrm{d}z} + A(z)\big)_R^k \big(C(z)\big)\, x\,. \tag{2.9}$$

For $k = N = 0$, we obviously have $L_f^0 h(x,z) = C(z)x$. Suppose that (2.9) holds for all $k \in \{0,\ldots,N\}$ for some $N \in \{0,\ldots,\ell-1\}$. Then

$$\begin{aligned}
L_f^{N+1} h(x,z) &= L_f\left(\left(\tfrac{\mathrm{d}}{\mathrm{d}z} + A\right)_R^N (C)\, x\right) \\
&= \left(\tfrac{\partial}{\partial x}\left(\left(\tfrac{\mathrm{d}}{\mathrm{d}z} + A\right)_R^N (C)\, x\right), \tfrac{\partial}{\partial z}\left(\left(\tfrac{\mathrm{d}}{\mathrm{d}z} + A\right)_R^N (C)\, x\right)\right)\begin{pmatrix} A\,x \\ 1 \end{pmatrix} \\
&= \left(\tfrac{\mathrm{d}}{\mathrm{d}z} + A\right)_R^N (C)\, A\, x + \tfrac{\partial}{\partial z}\left(\left(\tfrac{\mathrm{d}}{\mathrm{d}z} + A\right)_R^N (C)\, x\right) \\
&= \left(\tfrac{\mathrm{d}}{\mathrm{d}z} + A\right)_R^{N+1} (C)\, x\,.
\end{aligned}$$

This completes the proof of (2.9) and gives, for all $(x,z) \in \mathcal{U} \times \mathcal{T}$ and all $k \in \{0,\ldots,\ell\}$,

$$\begin{aligned}
L_g L_f^k h(x,z) &= \left(\tfrac{\partial}{\partial x}\left(\left(\tfrac{\mathrm{d}}{\mathrm{d}z} + A(z)\right)_R^k (C(z))x\right), \tfrac{\partial}{\partial z}\left(\left(\tfrac{\mathrm{d}}{\mathrm{d}z} + A(z)\right)_R^k (C(z))x\right)\right)\begin{pmatrix} B(z) \\ 0 \end{pmatrix} \\
&= \left(\tfrac{\mathrm{d}}{\mathrm{d}z} + A(z)\right)_R^k \big(C(z)\big) B(z)\,.
\end{aligned} \tag{2.10}$$

Finally, Theorem 2.6 is a consequence of (2.9) and (2.10). □

3 Normal form

In this section, we derive a normal form for time-varying linear systems (1.6). Theorem 2.6 may already indicate that the matrix function $\left(\frac{\mathrm{d}}{\mathrm{d}t} + A(\cdot)\right)_R^k \big(C(\cdot)\big)$, $k = 0,\ldots,r-1$ are candidates for a new basis, however, this potential basis needs to be completed. We introduce the following matrix functions which will serve to derive a time-varying linear transformation.

Let $r,\ell \in \mathbb{N}$ with $r \le \ell$. Consider the system (1.6) and define, for $r \in \mathbb{N}$ and all $t \in \mathbb{R}$,

$$\begin{aligned}
\mathcal{C}(t) &:= \begin{bmatrix} C(t) \\ \left(\frac{\mathrm{d}}{\mathrm{d}t} + A(t)\right)_R \big(C(t)\big) \\ \vdots \\ \left(\frac{\mathrm{d}}{\mathrm{d}t} + A(t)\right)_R^{r-1} \big(C(t)\big) \end{bmatrix} \in \mathbb{R}^{rm\times n} \\
\mathcal{B}(t) &:= \Big[B(t),\, \left(\tfrac{\mathrm{d}}{\mathrm{d}t} - A(t)\right)\big(B(t)\big),\, \ldots,\, \left(\tfrac{\mathrm{d}}{\mathrm{d}t} - A(t)\right)^{r-1}\big(B(t)\big)\Big] \in \mathbb{R}^{n\times rm} \\
\Gamma(t) &:= \left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A(t)\right)_R^{r-1} \big(C(t)\big) B(t) \in \mathbb{R}^{m\times m}\,.
\end{aligned}$$

The following proposition presents two more characterizations for (1.6) having relative degree r. They are rather technical but essential to design the coordinate transformation for the normal form.

Proposition 3.1 Let $r,\ell \in \mathbb{N}$ with $r \le \ell$ and $\mathcal{T} \subset \mathbb{R}$ be an open set. Then the following conditions are equivalent:

(i) The system (1.6) has relative degree r on $\mathcal{T}$.

(ii) (a) $\forall\, t \in \mathcal{T} \quad \forall\, (i,j) \in \{(i,j) \in \mathbb{N}_0^2 \mid 0 \le i+j \le r-2\}$:

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A(t)\right)_R^i \big(C(t)\big) \left(\tfrac{\mathrm{d}}{\mathrm{d}t} - A(t)\right)^j \big(B(t)\big) = 0_{m\times m}\,;$$

(b) $\forall\, t \in \mathcal{T} \quad \forall\, (i,j) \in \{(i,j) \in \mathbb{N}_0^2 \mid i+j = r-1\}\ :$

$$\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A(t)\right)_R^i \big(C(t)\big)\ \left(\tfrac{\mathrm{d}}{\mathrm{dt}} - A(t)\right)_R^j \big(B(t)\big)\ \in\ \mathrm{Gl}_m(\mathbb{R})\,.$$

(iii)

$$\forall\, t \in \mathcal{T}\ :\quad \mathcal{C}(t)\mathcal{B}(t) = \begin{bmatrix} 0 & & (-1)^{r-1}\Gamma(t) \\ & \cdot^{\cdot^{\cdot}} & \\ \Gamma(t) & & \divideontimes \end{bmatrix} \in \mathrm{Gl}_{rm}(\mathbb{R})\,.$$

The proof of Proposition 3.1 depends crucially on the following technical lemma.

Lemma 3.2 The linear time-varying system (1.6) satisfies, for all $i, j \in \mathbb{N}_0$ with $i + j \le \ell$ and all $t \in \mathbb{R}$,

$$\begin{aligned}\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A(t)\right)_R^i \big(C(t)\big) \cdot \left(\tfrac{\mathrm{d}}{\mathrm{dt}} - A(t)\right)^j \big(B(t)\big)& \\ = \tfrac{\mathrm{d}}{\mathrm{dt}}\Big[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A(t)\right)_R^i \big(C(t)\big) \cdot \left(\tfrac{\mathrm{d}}{\mathrm{dt}} - A(t)\right)^{j-1} \big(B(t)\big)\Big]& \\ - \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A(t)\right)_R^{i+1} \big(C(t)\big) \cdot \left(\tfrac{\mathrm{d}}{\mathrm{dt}} - A(t)\right)^{j-1} \big(B(t)\big),&\end{aligned} \tag{3.1}$$

and

$$\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A(t)\right)_R^i \big(C(t)\big)\ \left(\tfrac{\mathrm{d}}{\mathrm{dt}} - A(t)\right)^j \big(B(t)\big) = \sum_{\mu=0}^{j} (-1)^\mu \tbinom{j}{\mu} (\tfrac{\mathrm{d}}{\mathrm{dt}})^{j-\mu} \Big[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A(t)\right)_R^{i+\mu} \big(C(t)\big)\, B(t)\Big]\,. \tag{3.2}$$

Proof: The equality (3.1) follows from the calculation

$$\begin{aligned}
&\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^i (C) \cdot \left(\tfrac{\mathrm{d}}{\mathrm{dt}} - A\right)^j (B) \\
=\ & \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^i (C) \cdot \left(\tfrac{\mathrm{d}}{\mathrm{dt}} - A\right) \Big(\left(\tfrac{\mathrm{d}}{\mathrm{dt}} - A\right)^{j-1} (B)\Big) \\
=\ & \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^i (C) \cdot \Big[\tfrac{\mathrm{d}}{\mathrm{dt}}\Big(\left(\tfrac{\mathrm{d}}{\mathrm{dt}} - A\right)^{j-1} (B)\Big) - A\Big(\left(\tfrac{\mathrm{d}}{\mathrm{dt}} - A\right)^{j-1} (B)\Big)\Big] \\
=\ & \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^i (C) \cdot \Big(\cancel{\tfrac{\mathrm{d}}{\mathrm{dt}}\Big(\left(\tfrac{\mathrm{d}}{\mathrm{dt}} - A\right)^{j-1} (B)\Big)} - A\Big(\left(\tfrac{\mathrm{d}}{\mathrm{dt}} - A\right)^{j-1} (B)\Big)\Big) \\
& + \tfrac{\mathrm{d}}{\mathrm{dt}}\Big[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^i (C) \cdot \Big(\left(\tfrac{\mathrm{d}}{\mathrm{dt}} - A\right)^{j-1} (B)\Big)\Big] \\
& - \tfrac{\mathrm{d}}{\mathrm{dt}}\Big(\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^i (C)\Big) \cdot \Big(\left(\tfrac{\mathrm{d}}{\mathrm{dt}} - A\right)^{j-1} (B)\Big) - \Big(\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^i (C)\Big) \cdot \cancel{\tfrac{\mathrm{d}}{\mathrm{dt}}\Big(\left(\tfrac{\mathrm{d}}{\mathrm{dt}} - A\right)^{j-1} (B)\Big)} \\
=\ & \tfrac{\mathrm{d}}{\mathrm{dt}}\Big[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^i (C) \cdot \Big(\left(\tfrac{\mathrm{d}}{\mathrm{dt}} - A\right)^{j-1} (B)\Big)\Big] \\
& - \underbrace{\Big[\tfrac{\mathrm{d}}{\mathrm{dt}}\Big(\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^i (C)\Big) + \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^i (C) A\Big]}_{=\left(\frac{\mathrm{d}}{\mathrm{dt}} + A\right)_R\left(\left(\frac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^i (C)\right)} \cdot \left(\tfrac{\mathrm{d}}{\mathrm{dt}} - A\right)^{j-1} (B) \\
=\ & \tfrac{\mathrm{d}}{\mathrm{dt}}\Big[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^i (C) \cdot \Big(\left(\tfrac{\mathrm{d}}{\mathrm{dt}} - A\right)^{j-1} (B)\Big)\Big] - \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{i+1} (C) \cdot \left(\tfrac{\mathrm{d}}{\mathrm{dt}} - A\right)^{j-1} (B)\,.
\end{aligned}$$

We prove (3.2) by fixing $i \in \mathbb{N}_0$ and induction over $j = 0, \ldots, \ell - i$. For $j = 0$, (3.2) is obvious. Suppose that (3.2) holds for $j \leq \ell - i - 1$. Then, invoking (3.1), it follows that

$$
\begin{aligned}
&\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^i (C) \cdot \left(\tfrac{\mathrm{d}}{\mathrm{dt}} - A\right)^{j+1} (B) \\
&\quad = \tfrac{\mathrm{d}}{\mathrm{dt}} \left[\sum_{\mu=0}^{j} (-1)^\mu \tbinom{j}{\mu} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{j-\mu} \left[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{i+\mu} (C)\, B \right] \right] \\
&\qquad - \sum_{\mu=0}^{j} (-1)^\mu \tbinom{j}{\mu} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{j-\mu} \left[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{i+1+\mu} (C)\, B \right] \\
&\quad = \sum_{\mu=0}^{j} (-1)^\mu \tbinom{j}{\mu} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{j+1-\mu} \left[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{i+\mu} (C)\, B \right] \\
&\qquad - \sum_{\mu=1}^{j+1} (-1)^{\mu-1} \tbinom{j}{\mu-1} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{j-\mu+1} \left[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{i+1+\mu-1} (C)\, B \right] \\
&\quad = \tbinom{j}{0} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{j+1} \left[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{i+0} (C)\, B \right] \\
&\qquad + \sum_{\mu=1}^{j} \left[(-1)^\mu \tbinom{j}{\mu} - (-1)^{\mu-1} \tbinom{j}{\mu-1} \right] \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{j+1-\mu} \left[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{i+\mu} (C)\, B \right] \\
&\qquad - (-1)^{j+1-1} \tbinom{j}{j+1-1} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{j-j-1+1} \left[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{i+j+1} (C)\, B \right] \\
&\quad = (-1)^0 \tbinom{j}{0} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{j+1-0} \left[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{i+0} (C)\, B \right] \\
&\qquad + \sum_{\mu=1}^{j} (-1)^\mu \tbinom{j+1}{\mu} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{j+1-\mu} \left[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{i+\mu} (C)\, B \right] \\
&\qquad + (-1)^{j+1} \tbinom{j+1}{j+1} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{j+1-(j+1)} \left[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{i+j+1} (C)\, B \right] \\
&\quad = \sum_{\mu=0}^{j+1} (-1)^\mu \tbinom{j+1}{\mu} \left(\tfrac{\mathrm{d}}{\mathrm{dt}}\right)^{j+1-\mu} \left[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{i+\mu} (C)\, B \right].
\end{aligned}
$$

This completes the proof of the lemma. □

Proof of Proposition 3.1: The equivalence "(i) ⇔ (ii)" follows from (3.2) and (2.6), and the equivalence "(ii) ⇔ (iii)" follows from (3.2). □

The following corollary is a direct consequence of Proposition 3.1.

Corollary 3.3 If the linear time-varying system (1.6) has relative degree $r \in \mathbb{N}$ on some open set $\mathcal{T} \subset \mathbb{R}$, then the following hold:

(i) $\forall\, t \in \mathcal{T} \ : \ \operatorname{rk} \mathcal{C}(t) = rm$ and $\operatorname{rk} \mathcal{B}(t) = rm$;

(ii) the two sets of matrix functions $C(\cdot)$, $\left(\frac{\mathrm{d}}{\mathrm{dt}} + A(t)\right)_R \left(C(\cdot)\right), \ldots, \left(\frac{\mathrm{d}}{\mathrm{dt}} + A(t)\right)_R^{r-1} \left(C(\cdot)\right)$ and $B(\cdot)$, $\left(\frac{\mathrm{d}}{\mathrm{dt}} - A(\cdot)\right)\left(B(\cdot)\right)$, …, $\left(\frac{\mathrm{d}}{\mathrm{dt}} - A(\cdot)\right)^{r-1} \left(B(\cdot)\right)$ are both linearly independent over $\mathcal{T}$;

(iii) $rm \leq n$.

We are now in a position to design a time-varying linear basis transformation which will lead to a normal form.

Remark 3.4 Let $r, \ell \in \mathbb{N}$ with $r \leq \ell$ and $\mathcal{T} \subset \mathbb{R}$ be an open set. Suppose that the time-varying linear system (1.6) has relative degree $r \in \mathbb{N}$ on $\mathcal{T}$. By Corollary 3.3, the rows in $\mathcal{C}$ qualify as new basis but the basis needs to be completed. By Theorem 5.1, we may choose $T = [t_1, \ldots, t_n] \in \mathcal{C}^1(\mathcal{T}, \mathrm{Gl}_n(\mathbb{R}))$ such that

$$\begin{bmatrix} \mathcal{C} \\ 0 \end{bmatrix} T = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{C}^1(\mathcal{T}, \mathbb{R}^{n\times n}) \quad \text{with } F \in \mathcal{C}^1(\mathcal{T}, \mathrm{Gl}_{rm}(\mathbb{R})).$$

Defining

$$\mathcal{V} := [t_{rm+1}, \ldots, t_n] \in \mathcal{C}^1(\mathcal{T}, \mathbb{R}^{n\times(n-rm)})$$

we have

$$\forall\, t \in \mathcal{T} \;:\; \operatorname{im} \mathcal{V}(t) = \ker \mathcal{C}(t) \quad \text{and} \quad \operatorname{rk} \mathcal{V}(t)^T \mathcal{V}(t) = n - rm\,, \tag{3.3}$$

and writing

$$U := \begin{bmatrix} \mathcal{C} \\ \mathcal{N} \end{bmatrix} \in \mathcal{C}^1(\mathcal{T}, \mathbb{R}^{n\times n}) \quad \text{and} \quad \mathcal{N} := (\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T \left[I - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}\right] \in \mathcal{C}^1(\mathcal{T}, \mathbb{R}^{(n-rm)\times n})\,,$$

it follows from

$$\begin{bmatrix} \mathcal{C} \\ \mathcal{N} \end{bmatrix} \left[\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}, \mathcal{V}\right] = I_n,$$

that $U \in \mathcal{C}^1(\mathcal{T}, \mathrm{Gl}_n(\mathbb{R}))$ with inverse

$$U^{-1} = \left[\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}, \mathcal{V}\right] \in \mathcal{C}^1(\mathcal{T}, \mathrm{Gl}_n(\mathbb{R}))\,.$$

We are now in a position to derive the main result of this note, that is a normal form of the time-varying linear system (1.6).

Theorem 3.5
Let $r, \ell \in \mathbb{N}$ with $r \leq \ell$ and $\mathcal{T} \subset \mathbb{R}$ be an open set. Suppose the time-varying linear system (1.6) has relative degree r on $\mathcal{T}$ and choose $U \in C(\mathcal{T}, \mathrm{Gl}_n(\mathbb{R}))$, $\mathcal{V} \in \mathcal{C}^1(\mathcal{T}, \mathbb{R}^{n\times(n-r)})$, and $\mathcal{N} \in \mathcal{C}^1(\mathcal{T}, \mathbb{R}^{(n-rm)\times n})$ as in Remark 3.4. Then the coordinate transformation

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} := Ux, \qquad \xi(t) = (y(t)^T, \ldots, y^{(r-1)}(t)^T)^T \in \mathbb{R}^{rm},\ \eta(t) \in \mathbb{R}^{n-rm}$$

converts (1.6) on $\mathcal{T}$ into

$$\left.\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{dt}} \begin{pmatrix} \xi \\ \eta \end{pmatrix} &= \hat{A}(t) \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \hat{B}(t)\, u(t) \\ y(t) &= \hat{C}(t) \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} \end{aligned}\right\} \tag{3.4}$$

where

$$
\left.\begin{aligned}
\hat{A}(t) &= \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & I & 0 \\ R_1(t) & R_2(t) & \dots & R_r(t) & S(t) \\ P(t) & 0 & \dots & 0 & Q(t) \end{bmatrix}, \quad \hat{B}(t) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \Gamma \\ 0 \end{bmatrix}, \\
\hat{C}(t) &= \begin{bmatrix} I & 0 & \dots 0 \end{bmatrix},
\end{aligned}\right\} \tag{3.5}
$$

and

$$
\Gamma = \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{r-1}(C)B, \tag{3.6}
$$

$$
[R_1, \dots, R_r, S] = [0_{m\times(rm-m)}, I_m]\left[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{r}(\mathcal{C})\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}, \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{r}(\mathcal{C})\mathcal{V}\right] \tag{3.7}
$$

$$
= \left[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{r}(C)\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}, \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{r}(C)\mathcal{V}\right], \tag{3.8}
$$

$$
Q = \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R(\mathcal{N})\,\mathcal{V} \tag{3.9}
$$

$$
= -\,(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\left[\left(\tfrac{\mathrm{d}}{\mathrm{dt}} - A\right)\mathcal{V} - B\Gamma^{-1}\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{r}(C)\,\mathcal{V}\right], \tag{3.10}
$$

$$
P = \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R(\mathcal{N})\,\mathcal{B}\,(\mathcal{C}\mathcal{B})^{-1}\begin{bmatrix} I_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3.11}
$$

$$
= (-1)^{r-1}\,(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\,[\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C} - I]\left(\tfrac{\mathrm{d}}{\mathrm{dt}} - A\right)^r(B)\,\Gamma^{-1}. \tag{3.12}
$$

Proof: The special form of U and U^{-1} gives immediately $\hat{B} = UB$, $\hat{C}U = C$ with the special structure as shown in (3.5) and Γ as in (3.6); furthermore,

$$
\hat{A} = \left[UA + \dot{U}\right]U^{-1} = \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R(U)U^{-1}. \tag{3.13}
$$

In view of Remark 3.4, it remains to show (3.8)-(3.12) and that

$$
\begin{bmatrix} \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R(C) \\ \vdots \\ \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{r-1}(C) \\ \hline \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R^{r}(C) \\ \left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R(\mathcal{N}) \end{bmatrix} = \begin{bmatrix} 0 & I & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & 0 & I & 0 \\ \hline R_1 & R_2 & \dots & R_r & S \\ P_1 & P_2 & \dots & P_r & Q \end{bmatrix}\begin{bmatrix} \mathcal{C} \\ \mathcal{N} \end{bmatrix} \quad \text{with } P_2 = \dots = P_r = 0. \tag{3.14}
$$

We first show (3.14). Since equality of the upper blocks in (3.14) is immediate, it remains to show that

$$
\left(\tfrac{\mathrm{d}}{\mathrm{dt}} + A\right)_R(\mathcal{N})\,\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} = [P_1, 0, \dots, 0]. \tag{3.15}
$$

Writing

$$
\mathcal{C}\mathcal{B} = \left[\eta_1, \dots, \eta_r\right], \quad (\mathcal{C}\mathcal{B})^{-1} = \begin{bmatrix} \psi^1 \\ \vdots \\ \psi^r \end{bmatrix}, \quad \mathcal{B} = \left[\beta_1, \dots, \beta_r\right]
$$

we have

$$
\begin{aligned}
\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}\left(\tfrac{\mathrm{d}}{\mathrm{dt}}-A\right)(\mathcal{B}) &= \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}\left[\beta_2,\ldots,\beta_r,\left(\tfrac{\mathrm{d}}{\mathrm{dt}}-A\right)^r(B)\right] \\
&= \left[\beta_1,\ldots,\beta_r\right]\begin{bmatrix}\psi^1\\ \vdots \\ \psi^r\end{bmatrix}\left[\eta_2,\ldots,\eta_r,\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}\left(\tfrac{\mathrm{d}}{\mathrm{dt}}-A\right)^r(B)\right] \\
&= \left[\beta_1,\ldots,\beta_r\right]\begin{bmatrix}0 & 0 & \ldots & 0 & * \\ I & 0 & \ldots & 0 & * \\ 0 & I & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \\ 0 & \ldots & 0 & I & *\end{bmatrix} \\
&= \left[\beta_2,\ldots,\beta_r,\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}\left(\tfrac{\mathrm{d}}{\mathrm{dt}}-A\right)^r(B)\right],
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\left[\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}-I_n\right]\left(\tfrac{\mathrm{d}}{\mathrm{dt}}-A\right)(\mathcal{B}) \\
&\quad= \left[\beta_2,\ldots,\beta_r,(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}\left(\tfrac{\mathrm{d}}{\mathrm{dt}}-A\right)^r(B)\right]-\left[\beta_2,\ldots,\beta_r,\left(\tfrac{\mathrm{d}}{\mathrm{dt}}-A\right)^r(B)\right] \\
&\qquad= \left[0,\ldots,0,\left[\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}-I_n\right]\left(\tfrac{\mathrm{d}}{\mathrm{dt}}-A\right)^r(B)\right], \qquad (3.16)
\end{aligned}
$$

which, by invoking

$$
\begin{aligned}
&\left(\tfrac{\mathrm{d}}{\mathrm{dt}}+A\right)_R(\mathcal{N}) \\
&\quad= \tfrac{\mathrm{d}}{\mathrm{dt}}\left((\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\right)\left(I-\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}\right)+(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\left(A-\left(\tfrac{\mathrm{d}}{\mathrm{dt}}+A\right)_R\left(\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}\right)\right), \qquad (3.17)
\end{aligned}
$$

implies

$$
\begin{aligned}
\left(\tfrac{\mathrm{d}}{\mathrm{dt}}+A\right)_R(\mathcal{N})\mathcal{B} &= \Big[\tfrac{\mathrm{d}}{\mathrm{dt}}\left((\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\right)\left(I-\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}\right) \\
&\quad +(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\left(A-\left(\tfrac{\mathrm{d}}{\mathrm{dt}}+A\right)_R\left(\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}\right)\right)\Big]\mathcal{B} \\
&= (\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\left[A-\left(\tfrac{\mathrm{d}}{\mathrm{dt}}+A\right)_R\left(\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}\right)\right]\mathcal{B} \\
&= (\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\left[\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}-I_n\right]\left(\tfrac{\mathrm{d}}{\mathrm{dt}}-A\right)(\mathcal{B}) \\
&= (\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\left[0,\ldots,0,\left[\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}-I_n\right]\left(\tfrac{\mathrm{d}}{\mathrm{dt}}-A\right)^r(B)\right]. \qquad (3.18)
\end{aligned}
$$

Finally,

$$
(\mathcal{C}\mathcal{B})^{-1} = \begin{bmatrix} \divideontimes & & \Gamma^{-1} \\ & \iddots & \\ (-1)^{r-1}\Gamma^{-1} & & 0\end{bmatrix},
$$

applied to (3.18) yields (3.15), whence (3.14).
Next we prove (3.8). By (3.13),

$$
\left[R_1,\ldots,R_r,S\right] = [0_{m\times(rm-m)},I_m]\left(\tfrac{\mathrm{d}}{\mathrm{dt}}+A\right)_R(\mathcal{C})\ \left[\mathcal{B}(\mathcal{C}\mathcal{B})^{-1},\mathcal{V}\right],
$$

and (3.8) follows from the definition of $\mathcal{C}$.

We show (3.9) and (3.10). Equality (3.9) follows immediately from the normal form (3.4) and (3.5). To see equality (3.9), note that (3.17) yields

$$\begin{aligned}
&\left(\tfrac{\mathrm{d}}{\mathrm{dt}}+A\right)_R(\mathcal{N})\mathcal{V}\\
&= \Big[\tfrac{\mathrm{d}}{\mathrm{dt}}\left[(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\right]\left[I_n-\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}\right]-(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\tfrac{\mathrm{d}}{\mathrm{dt}}\left[\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}\right]\\
&\qquad +(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^TA-(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T(\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C})A\Big]\,\mathcal{V}\\
&= \tfrac{\mathrm{d}}{\mathrm{dt}}\left[(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\right]\big[\mathcal{V}-\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\underbrace{\mathcal{C}\mathcal{V}}_{=\,0}\big]\;+\;(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^TA\mathcal{V}\\
&\quad -(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\left[\tfrac{\mathrm{d}}{\mathrm{dt}}\left(\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}\right)\;+\;\left(\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}\right)A\right]\mathcal{V}\\
&= \underbrace{\tfrac{\mathrm{d}}{\mathrm{dt}}\left[(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\right]\mathcal{V}+(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^TA\mathcal{V}}_{=\left(\frac{\mathrm{d}}{\mathrm{dt}}+A\right)_R((\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V})\mathcal{V}}\\
&\quad -(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\Big[\dot{\mathcal{B}}(\mathcal{C}\mathcal{B})^{-1}\underbrace{\mathcal{C}\mathcal{V}}_{=\,0}+\mathcal{B}\tfrac{\mathrm{d}}{\mathrm{dt}}\left((\mathcal{C}\mathcal{B})^{-1}\right)\underbrace{\mathcal{C}\mathcal{V}}_{=\,0}+\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\dot{\mathcal{C}}\mathcal{V}+\left(\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}\right)A\mathcal{V}\Big],
\end{aligned}$$

which gives

$$\left(\tfrac{\mathrm{d}}{\mathrm{dt}}+A\right)_R(\mathcal{N})\mathcal{V}=\left(\tfrac{\mathrm{d}}{\mathrm{dt}}+A\right)_R\left((\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}\right)\mathcal{V}-(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\left(\tfrac{\mathrm{d}}{\mathrm{dt}}+A\right)_R(\mathcal{C})\mathcal{V}\,.$$

Invoking

$$\begin{aligned}
&\left(\tfrac{\mathrm{d}}{\mathrm{dt}}+A\right)_R\left((\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\right)\mathcal{V}\\
&= \tfrac{\mathrm{d}}{\mathrm{dt}}\left((\mathcal{V}^T\mathcal{V})^{-1}\right)\mathcal{V}^T\mathcal{V}+(\mathcal{V}^T\mathcal{V})^{-1}\tfrac{\mathrm{d}}{\mathrm{dt}}(\mathcal{V}^T)\mathcal{V}+(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^TA\mathcal{V}\\
&= -(\mathcal{V}^T\mathcal{V})^{-1}\tfrac{\mathrm{d}}{\mathrm{dt}}\left(\mathcal{V}^T\mathcal{V}\right)(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\mathcal{V}+(\mathcal{V}^T\mathcal{V})^{-1}\tfrac{\mathrm{d}}{\mathrm{dt}}(\mathcal{V}^T)\mathcal{V}+(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^TA\mathcal{V}\\
&= -(\mathcal{V}^T\mathcal{V})^{-1}\tfrac{\mathrm{d}}{\mathrm{dt}}(\mathcal{V}^T)\mathcal{V}-(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\tfrac{\mathrm{d}}{\mathrm{dt}}(\mathcal{V})+(\mathcal{V}^T\mathcal{V})^{-1}\tfrac{\mathrm{d}}{\mathrm{dt}}(\mathcal{V}^T)\mathcal{V}+(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^TA\mathcal{V}\\
&= -(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\left(\tfrac{\mathrm{d}}{\mathrm{dt}}-A\right)(\mathcal{V})\,,
\end{aligned}$$

we arrive at

$$\begin{aligned}
\left(\tfrac{\mathrm{d}}{\mathrm{dt}}+A\right)_R(\mathcal{N})\mathcal{V} &= -(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\left(\tfrac{\mathrm{d}}{\mathrm{dt}}-A\right)(\mathcal{V})\\
&\quad -(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\left(\tfrac{\mathrm{d}}{\mathrm{dt}}+A\right)_R(\mathcal{C})\,\mathcal{V}\\
&= -(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\left(\tfrac{\mathrm{d}}{\mathrm{dt}}-A\right)(\mathcal{V})\\
&\quad -(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\begin{bmatrix}0\\ \vdots\\ 0\\ \left(\frac{\mathrm{d}}{\mathrm{dt}}+A\right)_R^r(C)\,\mathcal{V}\end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= -(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\left(\tfrac{\mathrm{d}}{\mathrm{dt}}-A\right)(\mathcal{V})\\
&\quad -(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\mathcal{B}\begin{bmatrix} * & & \Gamma^{-1}\\ & \iddots & \\ (-1)^{r-1}\Gamma^{-1} & & 0\end{bmatrix}\begin{bmatrix}0\\ \vdots\\ 0\\ \left(\tfrac{\mathrm{d}}{\mathrm{dt}}+A\right)_R^r(C)\,\mathcal{V}\end{bmatrix}\\
&= -(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\left(\tfrac{\mathrm{d}}{\mathrm{dt}}-A\right)(\mathcal{V})\\
&\quad -(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\left[B,\ldots,\left(\tfrac{\mathrm{d}}{\mathrm{dt}}-A\right)^{r-1}(B)\right]\begin{bmatrix}\Gamma^{-1}\left(\tfrac{\mathrm{d}}{\mathrm{dt}}+A\right)_R^r(C)\,\mathcal{V}\\ 0\\ \vdots\\ 0\end{bmatrix}\\
&= -(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\left[\left(\tfrac{\mathrm{d}}{\mathrm{dt}}-A\right)(\mathcal{V}) \;-\; B\,\Gamma^{-1}\left(\tfrac{\mathrm{d}}{\mathrm{dt}}+A\right)_R^r(\mathcal{C})\,\mathcal{V}\right].
\end{aligned}
$$

This proves (3.10).

Finally, we show (3.11) and (3.12) for $P := P_1$. First, (3.14) together with Remark 3.4 yields

$$
\left(\tfrac{\mathrm{d}}{\mathrm{dt}}+A\right)_R(\mathcal{N})\;[\mathcal{B}(\mathcal{C}\mathcal{B})^{-1},\mathcal{V}] = [P,0\ldots,0] \tag{3.19}
$$

and hence (3.11) follows.

To see (3.12), note that (3.18) gives

$$
\begin{aligned}
&\left(\tfrac{\mathrm{d}}{\mathrm{dt}}+A\right)_R(\mathcal{N})\,\mathcal{B}\,(\mathcal{C}\mathcal{B})^{-1}\begin{bmatrix}I_m\\ 0\\ \vdots\\ 0\end{bmatrix}\\
&\qquad = (\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\left[0,\ldots,0,\left[\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}\;-\;I_n\right]\left(\tfrac{\mathrm{d}}{\mathrm{dt}}-A\right)^r(B)\right]\begin{bmatrix}*\\ \vdots\\ *\\ (-1)^{r-1}\,\Gamma^{-1}\end{bmatrix}
\end{aligned}
$$

which proves (3.12). This completes the proof of the theorem. □

As a direct consequence of Proposition 3.6 we obtain the following corollary.

Corollary 3.6 Under the assumptions of Theorem 3.5 and using the same notation, the normal form (3.4) may be written as

$$
\begin{aligned}
y^{(r)} &= \textstyle\sum_{i=1}^r R_i(t)\,y^{(i-1)} + S(t)\,\eta & &+ \Gamma(t)\,u(t)\\
\dot\eta &= Q(t)\,\eta & &+ P(t)\,y,
\end{aligned}
$$

and, if moreover (1.6) is time-invariant,

$$
\begin{aligned}
y^{(r)} &= \textstyle\sum_{i=1}^r R_i\,y^{(i-1)} + S\,\eta & &+ CA^{r-1}B\;u(t)\\
\dot\eta &= \mathcal{N}\,A\,\mathcal{V}\,\eta & &+ (-1)^{r-1}\,(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\,\mathcal{C}A^rB\,(CA^{r-1}B)^{-1}\;y\,.
\end{aligned}
$$

4 Zero dynamics

In the time-invariant, possibly nonlinear, case one can read off the zero dynamics from the normal form and, if the zero dynamics are asymptotically stable, a high-gain output derivative feedback controller may stabilize the system; see [3, Sec. 4.2]. In the time-varying case, an analogue cannot be expected unless severe restrictions on the time-variation of A, B, C are imposed. The reason is that, in general, neither the coordinate transformation U, designed in Remark 3.4, nor its inverse will be bounded. However, by Theorem 5.1 – an extended version of Doležal's Theorem – it is ensured that $\mathcal{V}$ and its left inverse $(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T$ are both bounded matrix functions and therefore the stability properties of the zero dynamics are equivalent to the stability properties of $\dot\eta = Q\eta$ (see (3.10)). To be precise, we first define the zero dynamics of a linear time-varying systems.

Definition 4.1 For any $r, \ell \in \mathbb{N}$ with $r \le \ell$ and $\mathcal{T} \subset \mathbb{R}$ an open set, the *zero dynamics of system* (1.6) *on* $\mathcal{T}$ are defined as the real vector space of trajectories

$$\mathcal{ZD}_{\mathcal{T}}(A,B,C) := \left\{ (x,u) \in \mathcal{C}^1(\mathcal{T},\mathbb{R}^n) \times \mathcal{C}^1(\mathcal{T},\mathbb{R}^m) \;\middle|\; \begin{array}{c} (x,u) \text{ solves (1.6)} \\ \text{with } y=0 \text{ on } \mathcal{T} \end{array} \right\}.$$

Also for time-varying systems, as known for time-invariant systems, the zero dynamics can be read off the normal form (3.4). This is shown in the following proposition. In fact, the zero dynamics can be parameterized.

Proposition 4.2 Let $r, \ell \in \mathbb{N}$ with $r \le \ell$ and $\mathcal{T} \subset \mathbb{R}$ an open set. Then for any system (1.6) with relative degree $r \le \ell$ on $\mathcal{T} \subset \mathbb{R}$ and normal form (3.4) the following holds.

$$\mathcal{ZD}_{\mathcal{T}}(A,B,C) = \left\{ \big(\mathcal{V}\eta, -\Gamma^{-1}S\eta\big) \in \mathcal{C}^1(\mathcal{T},\mathbb{R}^n) \times \mathcal{C}^1(\mathcal{T},\mathbb{R}^m) \;\middle|\; \dot\eta = Q\eta \right\}. \tag{4.1}$$

Proof: Set

$$\mathcal{Z} = \left\{ \big(\mathcal{V}\eta, -\Gamma^{-1}S\eta\big) \in \mathcal{C}^1(\mathcal{T},\mathbb{R}^n) \times \mathcal{C}^1(\mathcal{T},\mathbb{R}^m) \;\middle|\; \dot\eta = Q\eta \right\}.$$

"$\subseteq$": If $(x,u) \in \mathcal{ZD}_{\mathcal{T}}(A,B,C)$, then on $\mathcal{T}$ we have that $y = 0$ and so

$$\xi = (y^T, \dots, (y^{(r-1)})^T)^T = 0,$$

which yields, in view of (3.4),

$$x = \mathcal{V}\,\eta = U^{-1}\begin{pmatrix} 0 \\ \eta \end{pmatrix} \quad \text{and} \quad 0 = S\eta + \Gamma u\,,$$

and therefore, $(x,u) \in \mathcal{Z}$.
"$\supseteq$": If $(\tilde x, \tilde u) = \big(\mathcal{V}\eta, -\Gamma^{-1}S\eta\big) \in \mathcal{Z}$, then on $\mathcal{T}$ we have that

$$\tilde y := C\tilde x = C\mathcal{V}\eta = 0,$$

and so

$$\tilde\xi = (\tilde y^T, \dots, (\tilde y^{(r-1)})^T)^T = 0,$$

and therefore $\left(\begin{pmatrix} 0 \\ \eta \end{pmatrix}, \tilde u\right)$ solves the first equation in (3.4) with $\tilde y = 0$, and it follows that

$$(\tilde x, \tilde u) = \left(U^{-1}\begin{pmatrix} 0 \\ \eta \end{pmatrix}, \tilde u\right) = (\mathcal{V}\eta, \tilde u) \in \mathcal{ZD}_{\mathcal{T}}(A,B,C)\,.$$

This completes the proof of the proposition. $\square$

In the remainder of this section we study stability of the zero dynamics.

Definition 4.3 For any $t_0 \in \mathbb{R}$ and $\ell \in \mathbb{N}$ consider (1.6) on $\mathcal{T} = (t_0, \infty)$. Then the zero dynamics of (1.6) are called *asymptotically stable* if, and only if,

$$\forall\ (x,u) \in \mathcal{ZD}_{\mathcal{T}}(A,B,C)\ :\ \lim_{t\to\infty} x(t)\ =\ 0\,.$$

Although the time-varying coordinate transformation U which converts (1.6) to (3.4) may be unbounded or its inverse may be unbounded, we will show in the following theorem that – surprisingly – the zero dynamics of (1.6) are asymptotically stable if, and only if, $\dot{\eta} = Q\eta$ is an asymptotically stable system.

Theorem 4.4 Let $r, \ell \in \mathbb{N}$ with $r \leq \ell$, $t_0 \in \mathbb{R}$, and $\mathcal{T} = (t_0, \infty)$. Suppose the system (1.6) has relative degree $r \leq \ell$ on $\mathcal{T}$ and consider its normal form (3.4). Then the zero dynamics of (1.6) are asymptotically stable if, and only if, $\dot{\eta} = Q\eta$ is an asymptotically stable system.

Proof: It follows from Theorem 5.1 that $\mathcal{V}$ as defined in Remark 3.4 satisfies $\mathcal{V} \in \mathcal{L}^\infty(\mathcal{T}, \mathbb{R}^{n\times(n-rm)})$ and $(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T \in \mathcal{L}^\infty(\mathcal{T}, \mathbb{R}^{(n-rm)\times n})$. Now the claim of the theorem is a consequence of the fact that if $(x,u) \in \mathcal{ZD}_{\mathcal{T}}(A,B,C)$, then Proposition 4.2 yields

$$x = \mathcal{V}\eta \quad \text{and} \quad \eta = (\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T x\,.$$

□

5 Appendix: Doležal's Theorem re-revisited

Doležal's Theorem [1], which states convenient representations of range and kernels of time-varying matrices, has found numerous applications in systems theory and has been generalized and improved in various directions [2, 5, 6, 7]. In the following we give a generalization of [5, Theorem 2] which is tailored for the needs of the present paper.

Theorem 5.1
Let $\ell \in \mathbb{N}$, $M \in \mathcal{C}^\ell(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$ and suppose that there exists $r \in \{1,\dots,n\}$ such that, for all $t \geq 0$, $\operatorname{rk} M(t) = r$. Then there exists $T \in \mathcal{C}^\ell(\mathbb{R}_{\geq 0}, \mathrm{Gl}_n(\mathbb{R}))$ such that

$$\forall\ t \geq 0\ :\ M(t)\,T(t) = [F(t), 0_{n\times(n-r)}]\,, \tag{5.1}$$

$$\exists\ \beta > 0\quad \forall\ t \geq 0\ :\ \|T(t)\|\ \leq\ \beta\,, \tag{5.2}$$

$$\exists\ \varepsilon \in (0,1)\quad \forall\ t \geq 0\ :\ \varepsilon\ \leq\ T^T(t)T(t)\ \leq\ \frac{1}{\varepsilon}\,, \tag{5.3}$$

where, obviously, $\operatorname{rk} F(t) = r$ for all $t \geq 0$.
Moreover, for any $m \in \{1,\dots,n\}$ and partition

$$T(t) = [X(t), V(t)]\ ,\quad X(t) \in \mathbb{R}^{n\times(n-m)},\quad V(t) \in \mathbb{R}^{n\times m}\,, \tag{5.4}$$

we have

$$\exists\ \delta \in (0,1)\quad \forall\ t \geq 0\ :\ \delta\ \leq\ V^T(t)V(t)\ \leq\ \frac{1}{\delta}\,, \tag{5.5}$$

and $V \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times m})$ with left inverse $(V^TV)^{-1}V^T \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^{m\times n})$.

We preface the proof with the following lemma.

Lemma 5.2 Let $\ell \in \mathbb{N}$ and $P \in \mathcal{C}^\ell(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$ such that, for all $t \geq 0$, $P(t) = P(t)^T > 0$. Then the following statements are equivalent:

(i) $\exists\, \varepsilon \in (0,1) \quad \forall\, t \geq 0 \;:\; \varepsilon \leq \det P(t) \quad \text{and} \quad \|P(t)\| \leq 1/\varepsilon$

(ii) $\exists\, \delta \in (0,1) \quad \forall\, t \geq 0 \;:\; \delta \leq P(t) \leq 1/\delta \;,$

(iii) $\exists\, \delta \in (0,1) \quad \forall\, t \geq 0 \;:\; \delta \leq P(t)^{-1} \leq 1/\delta \;.$

Proof: Note that positivity of $P(t)$ yields

$$\forall\, t \geq 0 \quad \exists\, U(t) \in \mathbb{R}^{n\times n} \quad \text{and} \quad p_1(t), \ldots, p_n(t) > 0 \;:$$
$$U^T(t)U(t) = I_n \qquad \text{and} \qquad P(t) = U(t)\,\mathrm{diag}\big(p_1(t), \ldots, p_n(t)\big)\, U^T(t)\,, \tag{5.6}$$

and therefore,

$$\forall\; t \;\geq\; 0 \quad : \quad \det P(t) \;=\; \prod_{i=1}^n p_i(t) \quad \text{and} \quad \|\mathrm{diag}\big(p_1(t), \ldots, p_n(t)\big)\| \;=\; \|P(t)\|\,. \tag{5.7}$$

Hence we may conclude that

$$\begin{array}{lll}
\text{(i)} & \overset{(5.7)}{\iff} & \exists\, \delta \in (0,1) \quad \forall\, t \geq 0 \quad \forall\, i \in \{1, \ldots, n\} \;:\; \delta \leq p_i(t) \leq 1/\delta \\
 & \iff & \exists\, \delta \in (0,1) \quad \forall\, t \geq 0 \;:\; \delta \leq \mathrm{diag}\big(p_1(t), \ldots, p_n(t)\big) \leq 1/\delta \\
 & \overset{(5.6)}{\iff} & \text{(ii)} \\
 & \iff & \exists\, \delta \in (0,1) \quad \forall\, t \geq 0 \;:\; \delta \leq \mathrm{diag}\big(p_1(t), \ldots, p_n(t)\big) \leq 1/\delta \\
 & \iff & \exists\, \delta \in (0,1) \quad \forall\, t \geq 0 \quad \forall\, i \in \{1, \ldots, n\} \;:\; \delta \leq p_i(t) \leq 1/\delta \\
 & \iff & \exists\, \delta \in (0,1) \quad \forall\, t \geq 0 \quad \forall\, i \in \{1, \ldots, n\} \;:\; \delta \leq p_i(t)^{-1} \leq 1/\delta \\
 & \overset{(5.6)}{\iff} & \text{(iii)}.
\end{array}$$

This completes the proof of the lemma. □

Proof of Theorem 5.1: In [5, Theorem 2] it is shown that there exists $T \in \mathcal{C}^\ell(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$ such that

$$\exists\, \alpha \in (0,1) \quad \forall\, t \geq 0 \;:\; M(t)\,T(t) = [F(t), 0_{n\times(n-r)}] \text{ and } \alpha \leq |\det T(t)| \text{ and } \|T(t)\| \leq 1/\alpha\,.$$

This yields (5.1) and (5.2), and furthermore

$$\exists\, \alpha \in (0,1) \quad \forall\, t \geq 0 \;:\; \alpha^2 \leq \det T^T(t)\,T(t) \quad \text{and} \quad \big\|T^T(t)\,T(t)\big\| \leq 1/\alpha^2,$$

which implies, in view of Lemma 5.2, equation (5.3).

Let, for $m \in \{1, \ldots, n\}$, T be partitioned as in (5.4). The second inequality in (5.5) is a direct consequence of (5.2) and it remains to show the first inequality in (5.5). Seeking a contradiction, suppose that

$$\exists\, (\eta_i)_{i\in\mathbb{N}} \in \big(\mathcal{S}^{m-1}\big)^{\mathbb{N}} \quad \exists\, (t_i)_{i\in\mathbb{N}} \in (\mathbb{R}_{\geq 0})^{\mathbb{N}} \;:\; \lim_{i\to\infty} \eta_i^T V^T(t_i) V(t_i) \eta_i \;=\; 0\,,$$

where $\mathcal{S}^{m-1} := \{\eta \in \mathbb{R}^m | \ \|\eta\| = 1\}$. Then

$$
\begin{aligned}
0 &= \lim_{i\to\infty} \eta_i^T V^T(t_i) V(t_i) \eta_i \\
&= \lim_{i\to\infty} \left(0_{1\times(n-m)}, \eta_i^T\right) \begin{bmatrix} X^T(t_i)X(t_i) & X^T(t_i)V(t_i) \\ V^T(t_i)X(t_i) & V^T(t_i)V(t_i) \end{bmatrix} \begin{pmatrix} 0 \\ \eta_i \end{pmatrix} \\
&= \lim_{i\to\infty} \left(0_{1\times(n-m)}, \eta_i^T\right) T^T(t_i) T(t_i) \begin{pmatrix} 0_{n-m} \\ \eta_i \end{pmatrix},
\end{aligned}
$$

and this contradicts the first inequality in (5.3).

Finally, $V \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times m})$ is immediate from (5.2), and (5.5) together with Lemma 5.2 gives $V^T V \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^{m\times m})$, and therefore, again by Lemma 5.2, $(V^T V)^{-1} V^T \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^{m\times n})$. This completes the proof of the theorem. □

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