TRACKING WITH PRESCRIBED TRANSIENT BEHAVIOUR FOR NONLINEAR SYSTEMS OF KNOWN RELATIVE DEGREE*

ACHIM ILCHMANN†, EUGENE P. RYAN‡, AND PHILIP TOWNSEND‡

Abstract. Tracking of a reference signal (assumed bounded with essentially bounded derivative) is considered in the context of a class Σ_ρ of multi-input, multi-output dynamical systems, modelled by functional differential equations, affine in the control and satisfying the following structural assumptions: (i) arbitrary – but known – relative degree $\rho \geq 1$, (ii) the "high-frequency gain" is sign definite – but possibly of unknown sign. The class encompasses a wide variety of nonlinear and infinite-dimensional systems and contains (as a prototype subclass) all finite-dimensional, linear, m-input, m-output, minimum-phase systems of known strict relative degree. The first control objective is tracking, by the output y, with prescribed accuracy: given $\lambda > 0$ (arbitrarily small), determine a feedback strategy which ensures that, for every reference signal r and every system of class Σ_ρ, the tracking error $e = y - r$ is ultimately bounded by λ (that is, $\|e(t)\| < \lambda$ for all t sufficiently large). The second objective is guaranteed output transient performance: the tracking error is required to evolve within a prescribed performance funnel $\mathcal{F}_\varphi$ (determined by a function φ). Both objectives are achieved using a filter in conjunction with a feedback function of the tracking error, the filter states and the funnel parameter φ.

Keywords: Output feedback, nonlinear systems, functional differential equations, transient behaviour, tracking, high relative degree.

1. Introduction. In [5], a class of infinite-dimensional, m-input ($u(t) \in \mathbb{R}^m$), m-output ($y(t) \in \mathbb{R}^m$), nonlinear systems (with finite memory) given by a controlled functional differential equation of the form $\dot{y}(t) = g(p(t), (Ty)(t), u(t))$ is considered, where g is a continuous function, p represents a bounded disturbance and T is a causal operator with a bounded-input bounded-output property: an output feedback control structure is developed which ensures approximate asymptotic tracking, with prescribed transient behaviour, of any absolutely continuous bounded reference signal with essentially bounded derivative. Here, we extend these investigations to incorporate higher-order systems, affine in the control, of the form

$$y^{(\rho)}(t) = R_1\, y(t) + R_2\, y^{(1)}(t) + \cdots + R_\rho\, y^{(\rho-1)}(t) + g(p(t), (Ty)(t)) + \Gamma u(t) \qquad (1.1)$$

where $\rho \in \mathbb{N}$ is known, $y^{(i)}$ denotes the ith derivative of y and the matrix Γ is assumed to be sign definite (equivalently, $\langle v, \Gamma v\rangle = 0 \;\Leftrightarrow\; v = 0$).

In an early contribution by Miller and Davison [12], the attainment of prescribed transient behaviour is considered for a class of single-input, single-output, linear, minimum-phase systems with known high-frequency gain: a controller is introduced which guarantees the "error to be less than an (arbitrarily small) prespecified constant after an (arbitrarily small) prespecified period of time, with an (arbitrarily small) prespecified upper bound on the amount of overshoot." However, the controller is adaptive with non-decreasing gain k, invokes a piecewise-constant switching strategy, and is less flexible in its scope for shaping transient behaviour (in particular, an *a priori* bound on the initial data is required) when compared to the non-adaptive approach in [6].

*Based on work supported in part by the UK Engineering & Physical Sciences Research Council (GR/S94582/01).

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The results of this paper generalize the main ideas in [6], where tracking with prescribed transient behaviour is considered in a more restricted context of linear systems of known relative degree, subject to "mild" nonlinear perturbations: the generality of the operator T in (1.1) allows for a considerable diversity of nonlinear and infinite-dimensional effects, including delays and hysteresis phenomena. We implement a "backstepping" procedure in conjunction with a filter/pre-compensator in the construction of a non-adaptive controller. The backstepping procedure is akin to that of [17, 9, 12].

We briefly digress to review the literature on tracking and stabilization of high relative degree systems. Unless otherwise stated, all results relate to single-input, single-output systems. Bullinger and Allgöwer [1] introduce a high-gain observer in conjunction with an adaptive controller to achieve tracking with prescribed asymptotic accuracy $\lambda > 0$ (λ-tracking). This is achieved for a class of systems which are affine in the control, of known relative degree, and with affine linearly bounded drift term. Paper [17] considers linear minimum-phase systems with nonlinear perturbation; the control objective is (continuous) adaptive λ-tracking with non-decreasing gain. The class of allowable nonlinearities is considerably smaller than that of the present paper. Stabilization for systems of maximum relative degree in the so-called parametric strict feedback form is achieved in [18] via a piecewise constant adaptive switching strategy. Both these contributions use a backstepping procedure. Non-adaptive contributions are found in the work by Byrnes and Isidori [2] with extensions in [3]. They cover stabilization and tracking for a class of relative-degree-one nonlinear systems, with an exosystem, the positive orbits of which lie in a compact set: systems of higher relative degree are also considered, see in particular [2, (33)], and the authors state (without proof) that "these systems can reduced to systems of (relative degree 1) by means of the semiglobal back-stepping Lemma". The main result in [2, Proposition 7.1] pertains to practical tracking and applies high-gain principles in conjunction with an internal model: the multi-layered nature of the assumptions determining the system class makes it difficult to assess the overlap with the class considered in the present paper. Related investigations, based on high-gain properties and/or an internal model principle, can be found in [10, 13, 9]: we will have occasion to comment further on the latter in Section 3.1.3 below.

The paper is organized as follows. Sections 2 and 3 introduce the control objectives and the system class: Section 3.1 highlights several particular sub-classes. In Section 4, the control and feedback laws are constructed: an existence theorem for the resulting closed-loop system is provided in Section 4.3. Our main results on transient and asymptotic behaviour of the closed-loop are given in Section 5 and illustrated in an example in Section 6. All proofs are relegated to the Appendix.

We close this introduction with remarks on notation. Throughout, $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{C}_-$ denotes the open left half complex plane $\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda < 0\}$. The Euclidean inner product and induced norm on $\mathbb{R}^n$ are denoted by $\langle \cdot , \cdot \rangle$ and $\| \cdot \|$, respectively. The open ball of radius $\delta > 0$ centred at $x \in \mathbb{R}^n$ is denoted by $\mathbb{B}_\delta(x)$. For an interval $I \subset \mathbb{R}$, $C(I, \mathbb{R}^n)$ is the space of continuous functions $I \to \mathbb{R}^n$, $L^\infty(I, \mathbb{R}^n)$ is the space of essentially bounded measurable functions $x \colon I \to \mathbb{R}^n$ with norm $\|x\|_\infty := \text{ess-sup}_{t \in I} \|x(t)\|$, $L^1(I, \mathbb{R}^n)$ is the space of integrable functions $x \colon I \to \mathbb{R}^n$ with norm $\|x\|_1 := \int_I \|x(t)\| \mathrm{d}t < \infty$, $L^\infty_{\text{loc}}(I, \mathbb{R}^n)$ (respectively, $L^1_{\text{loc}}(I, \mathbb{R}^n)$) is the space of measurable, locally essentially bounded (respectively, locally integrable) functions $I \to \mathbb{R}^n$, and $W^{1,\infty}(I, \mathbb{R}^n)$ is the space of absolutely continuous functions $x \colon I \to \mathbb{R}^n$ with $x, \dot{x} \in L^\infty(I, \mathbb{R}^n)$. The spectrum of $A \in \mathbb{R}^{n \times n}$ is denoted by $\operatorname{spec}(A)$.

2. Control objectives and the performance funnel. There are two control objectives: (i) approximate tracking, by the output, of reference signals $r \in \mathcal{R} := W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$. In particular, for arbitrary $\lambda > 0$, we seek an output feedback strategy which ensures that, for every $r \in \mathcal{R}$, the closed-loop system has bounded solution and the tracking error $e(t) = y(t) - r(t)$ is ultimately bounded by λ (that is, $\|e(t)\| \le \lambda$ for all t sufficiently large), and (ii) prescribed transient behaviour of the tracking error.

Both objectives are captured in the concept of a performance funnel

$$\mathcal{F}_\varphi := \left\{ (t,e) \in \mathbb{R}_+ \times \mathbb{R}^m \,\middle|\, \varphi(t)\|e\| < 1 \right\}$$

associated with a function φ (the reciprocal of which determines the funnel boundary) of the following class

$$\Phi := \left\{ \varphi \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}) \,\middle|\, \varphi(0) = 0, \quad \varphi(s) > 0 \text{ for all } s > 0 \text{ and } \liminf_{s\to\infty} \varphi(s) > 0 \right\}.$$

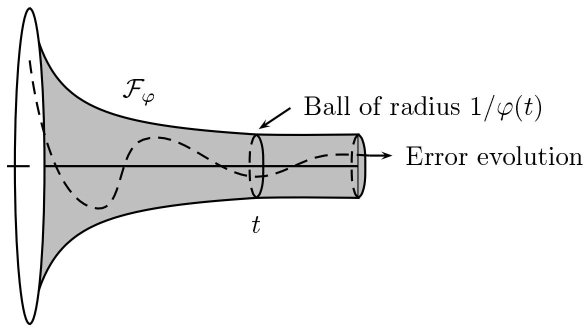


Figure 1: Prescribed performance funnel $\mathcal{F}_\varphi$.

The aim is an output feedback strategy ensuring that, for every reference signal $r \in \mathcal{R}$, the tracking error $e = y - r$ evolves within the funnel $\mathcal{F}_\varphi$. For example, if $\liminf_{t\to\infty} \varphi(t) \ge 1/\lambda$, then evolution within the funnel ensures that the first control objective is achieved. If φ is chosen as the function $t \mapsto \min\{t/\tau, 1\}/\lambda$, then evolution within the funnel ensures that the prescribed tracking accuracy $\lambda > 0$ is achieved within the prescribed time $\tau > 0$. The feedback structure incorporates a filter and essentially exploits an intrinsic high-gain property of the system/filter interconnection to ensure that, if $(t, e(t))$ approaches the funnel boundary, then an appropriately generated gain attains values sufficiently large to preclude boundary contact.

3. Class of systems. We subsume (1.1) in the following

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + f(p(t), (Ty)(t), x(t)) + Bu(t), \\ y(t) &= Cx(t), \\ x|_{[-h,0]} &= x^0 \in C([-h,0], \mathbb{R}^{\rho m}), \end{aligned}\right\} \tag{3.1}$$

$$A = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I \\ R_1 & R_2 & \cdots & R_{\rho-1} & R_\rho \end{bmatrix} \in \mathbb{R}^{\rho m \times \rho m}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \Gamma \end{bmatrix} \in \mathbb{R}^{\rho m \times m}, \tag{3.2}$$

$$C = \quad [I \,\vdots\, 0 \,\vdots\, \cdots \,\vdots\, 0 \,\vdots\, 0] \in \mathbb{R}^{m \times \rho m}, \quad f\colon \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^{\rho m} \to \mathbb{R}^{\rho m} \text{ continuous.} \tag{3.3}$$

Observe that $\Gamma = CA^{\rho-1}B$. In the special case wherein f is given by

$$f(p,w,x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ g(p,w) \end{bmatrix},$$

it is clear that (1.1) and (3.1) are equivalent. Next, we define the class of operators T allowable in (3.1).

DEFINITION 3.1. **(Operator class $\mathcal{T}_h$)**
Let $h \geq 0$. An operator T is said to be of class $\mathcal{T}_h$ if, and only if, for some $l, q \in \mathbb{N}$, the following hold.
(i) $T\colon C([-h,\infty),\mathbb{R}^l) \to L^\infty_{\rm loc}(\mathbb{R}_+,\mathbb{R}^q)$.
(ii) *For every $\delta > 0$, there exists $\Delta > 0$ such that, for all $\zeta \in C([-h,\infty),\mathbb{R}^l)$,*

$$\sup_{t\in[-h,\infty)} \|\zeta(t)\| \leq \delta \quad \Longrightarrow \quad \|(T\zeta)(t)\| \leq \Delta \quad \textit{for almost all } t \geq 0 .$$

(iii) *For all $t \in \mathbb{R}_+$, the following hold:*
(a) *for all $\zeta, \psi \in C([-h,\infty),\mathbb{R}^l)$,*

$$\zeta(\cdot) \equiv \psi(\cdot) \ \textit{on } [-h,t] \quad \Longrightarrow \quad (T\zeta)(s) = (T\psi)(s) \ \ \textit{for almost all } s \in [0,t];$$

(b) *for all continuous functions $\beta\colon [-h,t] \to \mathbb{R}^l$, there exist $\tau, \delta, c > 0$ such that, for all $\zeta, \psi \in C([-h,\infty),\mathbb{R}^l)$ with $\zeta|_{[-h,t]} = \beta = \psi|_{[-h,t]}$ and $\zeta(s), \psi(s) \in \mathbb{B}_\delta(\beta(t))$ for all $s \in [t, t+\tau]$,*

$$\text{ess-sup}_{s\in[t,t+\tau]}\|(T\zeta)(s) - (T\psi)(s)\| \ \leq \ c \sup_{s\in[t,t+\tau]} \|\zeta(s) - \psi(s)\| .$$

REMARK 3.2. Property (ii) is a bounded-input, bounded-output assumption on the operator T. Property (iii)(a) is a natural assumption of causality. Property (iii)(b) is a technical assumption of local Lipschitz type which is used in establishing well-posedness of the closed-loop system (defined later in Subsection 4.3).

We are now in a position to make precise the system class.

DEFINITION 3.3. **(System class $\boldsymbol{\Sigma}_\rho$)**
For $\rho \in \mathbb{N}$, Σ_ρ is the class of m-input, m-output systems (A, B, C, f, p, T, h) of the form (3.1)*, where $h \geq 0$ quantifies the memory of the system, A, B and C are structured as in* (3.2)-(3.3) *and satisfy*
(A1) *sign-definite high-frequency gain: $\Gamma = CA^{\rho-1}B$ is either positive definite or negative definite (equivalently, $\langle v, \Gamma v\rangle = 0 \ \Leftrightarrow \ v = 0$).*
The functions f, p and operator T are such that
(A2) $p \in L^\infty(\mathbb{R}_+,\mathbb{R}^m)$,
(A3) *for some $q \in \mathbb{N}$, $T\colon C([-h,\infty),\mathbb{R}^m) \to L^\infty_{\rm loc}(\mathbb{R}_+,\mathbb{R}^q)$ is of class $\mathcal{T}_h$,*
(A4) *$f\colon \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^{\rho m} \to \mathbb{R}^{\rho m}$ is continuous and, for all non-empty compact sets $P \subset \mathbb{R}^m$, $W \subset \mathbb{R}^q$ and $Y \subset \mathbb{R}^{\rho m}$, there exists a constant $c_0 = c_0(P,W,Y) > 0$ such that $\|f(p,w,x)\| \leq c_0$ for all $(p,w,x) \in P \times W \times \{v \in \mathbb{R}^{\rho m} \mid Cv \in Y\}$.*

REMARK 3.4.
(i) Due to the presence of the nonlinear function f, the (vector) relative degree of (3.1) at some point $x^0 \in \mathbb{R}^{\rho m}$ may not be defined, see [7, pp. 137 and 220]. However, if $f \equiv 0$, then it follows from Assumption (A1) that the vector relative degree of the linear system (3.1) is $(\rho,\ldots,\rho) \in \mathbb{R}^m$ at each point $x^0 \in \mathbb{R}^{\rho m}$ and, in particular,

$$CA^iB = 0 \quad \text{for } i = 1,...,\rho-2 \text{ and } \ \Gamma = CA^{\rho-1}B \ \text{ is invertible.} \tag{3.4}$$

The linear system (A, B, C) is said to have *strict relative degree* ρ if, and only if, (3.4) holds. Note that Assumption (A1) requires the strengthened assumption that $CA^{\rho-1}B$ is either positive definite or negative definite. In the multi-input, multi-output case, (A1) is rather restrictive. By contrast, in the single-input, single-output case, the assumption of sign definiteness is redundant and (A1) is simply equivalent to positing that the relative degree of the linear triple (A, B, C) is known.
(ii) Recall that a linear system (A, B, C) is said to be *minimum phase* if, and only if,

$$\det \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} \neq 0 \qquad \text{for all } s \in \mathbb{C} \text{ with } \mathrm{Re}(s) \geq 0. \tag{3.5}$$

Due to the structure of the matrices A, B and C in (3.2)-(3.3) and Assumption (A1), (A, B, C) is minimum phase.
(iii) With reference to Figure 2, the system (3.1) can be thought of as the interconnection of two blocks. The dynamical system represented by block Λ_1, which can be influenced directly by the system control u, is also driven by the output w from the dynamic block Λ_2, as shown in Figure 2. The block Λ_2 can be considered as a causal operator mapping the system output y to w (an internal quantity, unavailable for feedback purposes); it allows for infinite-dimensional (e.g. delays, diffusions) and hysteresis (e.g. backlash) effects, some examples of which are given in Section 3.1.

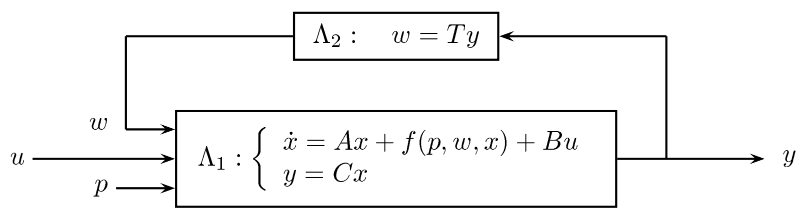


Figure 2: System of class Σ_ρ.

3.1. Sub-classes of Σ_ρ.

3.1.1. Finite-dimensional linear prototype. For motivational purposes, we first examine a prototype linear system and show that all finite-dimensional linear systems of this form are incorporated in the class Σ_ρ. Consider an m-input, m-output linear system of the form

$$\dot{w}(t) = \tilde{A}\, w(t) + \tilde{B}\, u(t), \quad w(0) = w^0 \in \mathbb{R}^n, \quad y(t) = \tilde{C}\, w(t), \tag{3.6}$$

with strict relative degree $\rho \geq 1$, $\tilde{A} \in \mathbb{R}^{n \times n}$, $\tilde{B} \in \mathbb{R}^{n \times m}$, $\tilde{C} \in \mathbb{R}^{m \times n}$, $n \geq \rho m$ and positive-definite or negative-definite high-frequency gain $\tilde{C}\tilde{A}^{\rho-1}\tilde{B}$. To show that the system (3.6) belongs to the class Σ_ρ, we present the following lemma, a proof of which can be found in the Appendix.

Lemma 3.5. *Consider a linear system of the form* (3.6) *with strict relative degree* $\rho \in \mathbb{N}$. *Define*

$$\mathcal{C} := \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \vdots \\ \tilde{C}\tilde{A}^{\rho-1} \end{bmatrix} \in \mathbb{R}^{\rho m \times n}, \qquad \mathcal{B} := [\tilde{B} \,\vdots\, \tilde{A}\tilde{B} \,\vdots\, \cdots \,\vdots\, \tilde{A}^{\rho-1}\tilde{B}] \in \mathbb{R}^{n \times \rho m}$$

and let $\mathcal{V} \in \mathbb{R}^{n\times(n-\rho m)}$ *be such that* $\operatorname{im} \mathcal{V} = \ker \mathcal{C}$. *Then*
(i) $\mathbb{R}^n = \ker \mathcal{C} \oplus \operatorname{im} \mathcal{B}$;
(ii) *the matrix*

$$\mathcal{U} = \begin{bmatrix} \mathcal{C} \\ \mathcal{N} \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad \textit{where} \quad \mathcal{N} = (\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T[I - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}] \in \mathbb{R}^{(n-\rho m)\times n},$$

is invertible, with inverse $\mathcal{U}^{-1} = [\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \vdots \mathcal{V}]$, *and the triple*

$$(\hat{A}\,,\,\hat{B}\,,\,\hat{C}) := \big(U\tilde{A}U^{-1},\,U\tilde{B}\,,\,\tilde{C}U^{-1}\big) \tag{3.7}$$

has the following structure (wherein I *and* 0 *denote the* $m\times m$ *identity matrix and zero matrix, respectively)*

$$\hat{A} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & I & 0 \\ R_1 & R_2 & \cdots & R_{\rho-1} & R_\rho & S \\ P & 0 & \cdots & 0 & 0 & Q \end{bmatrix}, \ \hat{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \Gamma \\ 0 \end{bmatrix}, \ \hat{C} = [I \vdots 0 \vdots \cdots \vdots 0 \vdots 0 \vdots 0], \tag{3.8}$$

with $[R_1 \vdots \cdots \vdots R_\rho \vdots S] = \tilde{C}\tilde{A}^\rho\mathcal{U}^{-1}$, $\Gamma = \tilde{C}\tilde{A}^{\rho-1}\tilde{B}$, $P = \mathcal{N}\tilde{A}^\rho\tilde{B}\Gamma^{-1}$, *and* $Q = \mathcal{N}\tilde{A}\mathcal{V}$;

(iii) *if the system* (3.6) *is minimum phase, then* $\operatorname{spec}(Q) \subset \mathbb{C}_-$.

We remark that, in the case $\rho = 1$, (3.8) is to be interpreted as

$$\hat{A} = \begin{bmatrix} R_1 & S \\ P & Q \end{bmatrix}, \ \hat{B} = \begin{bmatrix} \Gamma \\ 0 \end{bmatrix}, \ \hat{C} = [I \vdots 0]. \tag{3.9}$$

Invoking the similarity transformation (3.7)-(3.8) and writing $x^0 := \mathcal{C}w^0$, $z^0 := \mathcal{N}w^0$, $x(t) := \mathcal{C}w(t)$, it is readily verified that system (3.6) is equivalent to

$$\left.\begin{aligned} &\dot{x}(t) = Ax(t) + f(p(t),(Ty)(t)) + Bu(t), \quad x(0) = x^0, \\ &y(t) = Cx(t), \end{aligned}\right\} \tag{3.10}$$

where A, B and C are as in (3.2)-(3.3), $p\colon t \mapsto S(\exp Qt)z^0$, T is the linear operator given by

$$(Ty)(t) = S\left(\int_0^t \exp(Q(t-s))Py(s)\mathrm{d}s\right), \quad t \ge 0,$$

and the function $f\colon \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is given by $f(\alpha,\beta) := \alpha + \beta$.

If (3.6) has sign definite high-frequency gain, then $\tilde{C}\tilde{A}^{\rho-1}\tilde{B} = \Gamma = CA^{\rho-1}B$ is either positive definite or negative definite and hence Assumption (A1) is satisfied. If we assume that (3.6) has the minimum-phase property, then by Lemma 3.5 (iii), Q has spectrum in $\mathbb{C}_-$: it follows that $p \in L^\infty(\mathbb{R}_+,\mathbb{R}^m)$ and T belongs to the class of operators $\mathcal{T}_0$ and so Assumptions (A2) and (A3) are satisfied. Assumption (A4) is trivially satisfied. Therefore, the system class Σ_ρ contains all m-input, m-output, finite-dimensional, linear, minimum-phase systems of strict relative degree ρ with sign-definite high-frequency gain.

3.1.2. Infinite-dimensional linear systems. The finite-dimensional class of systems of the form (3.7) can be extended to infinite dimensions by reinterpreting the operators Q, P and S as the generating operators of a regular linear system (regular in the sense of [16]). In the infinite-dimensional setting, Q is assumed to be the generator of a strongly continuous semigroup $\mathbf{S} = (\mathbf{S}_t)_{t\in\mathbb{R}_+}$ of bounded linear operators and a Hilbert space X with norm $\|\cdot\|_X$. Let X_1 denote the space $\mathrm{dom}(Q)$ endowed with the graph norm and let X_{-1} denote the completion of X with respect to the norm $\|z\|_{-1} = \|(s_0 I - Q)^{-1}z\|_X$, where s_0 is any fixed element of the resolvent set of Q. Then P is assumed to be a bounded linear operator from $\mathbb{R}^m$ to X_{-1} and S is assumed to be a bounded linear operator from X_1 to $\mathbb{R}^m$. Assuming that the semigroup $\mathbf{S}$ is exponentially stable and that S extends to a bounded linear operator (again denoted by S) from X to $\mathbb{R}^m$, then the operator T given by

$$(Ty)(t) := S\left(\int_0^t \mathbf{S}_{t-s} P y(s)\,\mathrm{d}s\right)$$

is of class $\mathcal{T}_0$ (see [14] for details) and, writing $p(t) := S\,\mathbf{S}_t z^0$, we again arrive at structure of (3.10).

3.1.3. Nonlinear systems. In [9, eqn. (1)] the following class of systems is studied

$$\left.\begin{aligned}
\dot{x}_1(t) &= x_2(t) + f_1(w(t), y(t))\\
&\ \vdots\\
\dot{x}_{\rho-1}(t) &= x_\rho(t) + f_{\rho-1}(w(t), y(t))\\
\dot{x}_\rho(t) &= \gamma\, u(t) + f_\rho(w(t), y(t))\\
\dot{w}(t) &= q(w(t), y(t))\\
y(t) &= x_1(t)\\
(x_1(0), \ldots, x_\rho(0), w(0)) &= (x_1^0, \ldots, x_\rho^0, w^0)
\end{aligned}\right\} \tag{3.11}$$

where $\gamma \in \mathbb{R}\setminus\{0\}$, $q\colon \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^p$ and $f_i\colon \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, \rho$, are locally Lipschitz functions. Denote, by T, the mapping $y \mapsto w$ induced by the subsystem $\dot{w} = q(w, y)$ with initial condition $w(0) = w^0$. Then (3.11) is equivalent to (3.1) (with $h = 0$ and $m = 1$). Moreover, if we assume that the subsystem $\dot{w} = q(w, y)$ is input-to-state stable (ISS), then, as shown in [4, Section 2.3], the operator T is of class $\mathcal{T}_0$, in which case system (3.11), interpreted in its equivalent form (3.1), is of class Σ_ρ.

We remark that, in [9, eqn. (1)], an assumption of *integral* input-to-state stability (iISS) (strictly weaker than our assumption of ISS) is imposed on the subsystem $\dot{w} = q(w, y)$. In this respect, the full generality of the system class in [9] is not captured by the class considered in the present paper.

3.1.4. Nonlinear delay systems. Let functions $\mathcal{G}_i\colon \mathbb{R}\times\mathbb{R}^l \to \mathbb{R}^q : (t,\zeta) \mapsto \mathcal{G}_i(t,\zeta)$, $i = 0, \ldots, n$ be measurable in t and locally Lipschitz in ζ uniformly with respect to t: precisely, (i) for each fixed ζ, $\mathcal{G}_i(\cdot,\zeta)$ is measurable and (ii) for every compact $\mathcal{K} \subset \mathbb{R}^l$ there exists a constant c such that

$$\|\mathcal{G}_i(t,\zeta) - \mathcal{G}_i(t,\psi)\| \le c\,\|\zeta - \psi\| \quad \text{for almost all } t \text{ and for all } \zeta, \psi \in \mathcal{K}\,.$$

For $i = 0, \ldots n$, let $h_i \in \mathbb{R}_+$ and define $h := \max_i h_i$. For $\zeta \in C([-h,\infty), \mathbb{R}^l)$, let

$$(T\zeta)(t) := \int_{-h_0}^0 \mathcal{G}_0(s, \zeta(t+s))\,\mathrm{d}s + \sum_{i=1}^n \mathcal{G}_i(t, \zeta(t - h_i)) \quad \text{for all } t \ge 0\,.$$

The operator T, so defined, is of class $\mathcal{T}_h$: for details see [14].

3.1.5. Systems with hysteresis. A general class of hysteresis operators, which includes many physically motivated hysteretic effects, is discussed in [11]. Examples of such operators include backlash hysteresis, elastic-plastic hysteresis and Preisach operators. In [5], it is pointed out that these operators are of class $\mathcal{T}_0$. For illustration, we describe a particular example of a hysteresis operator.

Backlash hysteresis: Consider a one-dimensional mechanical link consisting of two components, denoted I and II (of width $2a$) and illustrated in Figure 4a.

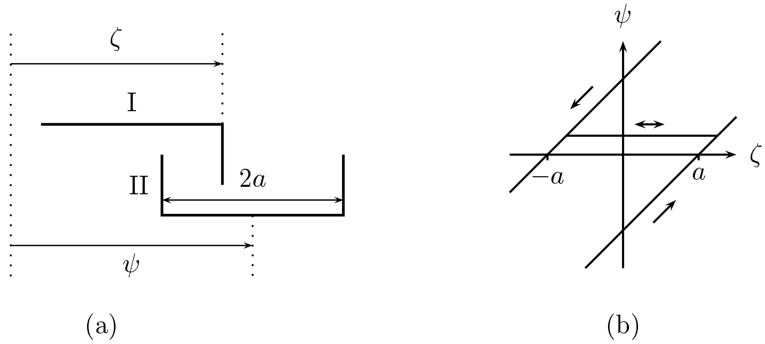


Figure 3: Backlash hysteresis

The displacements of each part (with respect to some fixed datum) at time $t \geq 0$ are given by $\zeta(t)$ and $\psi(t)$ with $|\zeta(t) - \psi(t)| \leq a$ for all t, and $\psi(0) := \zeta(0) + b$ for some pre-specified $b \in [-a, a]$. Within the link there is mechanical play: that is to say the position $\psi(t)$ of II remains constant as long as the position $\zeta(t)$ of I remains within the interior of II. Thus, assuming continuity of ζ, we have $\dot{\psi}(t) = 0$ whenever $|\zeta(t) - \psi(t)| < a$. Given a continuous input $\zeta \in C(\mathbb{R}_+, \mathbb{R})$, describing the evolution of the position of I, denote the corresponding position of II by $\psi = T\zeta$. The operator T, (in effect we define a family $T_{a,b}$ of operators parameterized by $a > 0$ and $b \in [-a, a]$) so defined, is known as *backlash* or *play* and is of class $\mathcal{T}_0$.

4. The control. Let Assumption (A1) hold, with relative degree $\rho \geq 2$; the relative degree 1 case will be treated separately.

4.1. Filter. Fix $\mu > 0$ (arbitrarily) and introduce the filter

$$\begin{aligned} \dot{\xi}_i(t) &= -\mu\, \xi_i(t) + \xi_{i+1}, & \xi_i(0) &= \xi_i^0 \in \mathbb{R}^m, & i &= 1, \ldots, \rho-2, \\ \dot{\xi}_{\rho-1}(t) &= -\mu\, \xi_{\rho-1}(t) + u(t), & \xi_{\rho-1}(0) &= \xi_{\rho-1}^0 \in \mathbb{R}^m, \end{aligned}$$

which, on writing (wherein I and 0 denote the $m \times m$ identity and zero matrices)

$$\xi(t) = \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \\ \vdots \\ \xi_{\rho-2}(t) \\ \xi_{\rho-1}(t) \end{bmatrix}, \quad F = \begin{bmatrix} -\mu I & I & 0 & \cdots & 0 & 0 \\ 0 & -\mu I & I & \cdots & 0 & 0 \\ 0 & 0 & -\mu I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\mu I & I \\ 0 & 0 & 0 & \cdots & 0 & -\mu I \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix},$$

may be expressed as

$$\left.\begin{aligned} \dot{\xi}(t) &= F\xi(t) + Gu(t), & \xi(0) &= \xi^0 \in \mathbb{R}^{(\rho-1)m}, \\ \xi_1(t) &= H\xi(t), & H &:= [I \vdots 0 \vdots 0 \vdots \cdots \vdots 0 \vdots 0]. \end{aligned}\right\} \tag{4.1}$$

4.2. Feedback. Let $\nu\colon \mathbb{R}\to\mathbb{R}$ be any C^∞ function with the property that, for some increasing unbounded sequence $(k_j)\subset\mathbb{R}_+$,

$$s(\Gamma)\nu(k_j)\to-\infty \quad \text{as } j\to\infty\,, \quad \text{where} \quad s(\Gamma):=\begin{cases}+1\,, & \Gamma \text{ positive definite,}\\ -1\,, & \Gamma \text{ negative definite.}\end{cases} \tag{4.2}$$

Let $\alpha\colon[0,1)\to\mathbb{R}_+$ be a C^∞ unbounded injection such that, for some $\delta>0$, its derivative satisfies $\alpha'(s)\ge\delta$ for all $s\in[0,1)$; for example $\alpha\colon s\mapsto 1/(1-s)$. Introduce the projections

$$\pi_i\colon \mathbb{R}^{(\rho-1)m}\to\mathbb{R}^{im},\quad \xi=(\xi_1,\ldots,\xi_{\rho-1})\mapsto(\xi_1,\ldots,\xi_i),\qquad i=1,\ldots,\rho-1,$$

and define the C^∞ function

$$\gamma_1\colon\mathbb{R}\times\mathbb{R}^m\to\mathbb{R}^m,\qquad (k,e)\mapsto\gamma_1(k,e):=-\nu(k)e\,, \tag{4.3}$$

with derivative (Jacobian matrix function) $D\gamma_1$. Next, for $i=2,\ldots,\rho-1$, define the C^∞ function $\gamma_i\colon\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^{(i-1)m}\to\mathbb{R}^m$ by the recursion

$$\begin{aligned}\gamma_i(k,e,\pi_{i-1}\xi):=&\gamma_{i-1}(k,e,\pi_{i-2}\xi)\\ &+\|D\gamma_{i-1}(k,e,\pi_{i-2}\xi)\|^2\,(\alpha'(\varphi^2||e||^2))^2\,(1+\|\pi_{i-1}\xi\|^2)\\ &\times\Big(\mu^{2-i}\xi_{i-1}+\gamma_{i-1}(k,e,\pi_{i-2}\xi)\Big)\,,\end{aligned} \tag{4.4}$$

wherein we adopt the notational convention $\gamma_1(k,e,\pi_0\xi):=\gamma_1(k,e)$. Finally, define the C^∞ function $\gamma_\rho\colon\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^{(\rho-1)m}\to\mathbb{R}^m$ as follows

$$\begin{aligned}\gamma_\rho(k,e,\xi):=&\mu^{\rho-1}\gamma_{\rho-1}(k,e,\pi_{\rho-2}\xi)\\ &+\mu^{\rho-1}\|D\gamma_{\rho-1}(k,e,\pi_{\rho-2}\xi)\|^2\,(\alpha'(\varphi^2||e||^2))^2\,(1+\|\xi\|^2)\\ &\times\Big(\mu^{2-\rho}\xi_{\rho-1}+\gamma_{\rho-1}(k,e,\pi_{\rho-2}\xi)\Big).\end{aligned} \tag{4.5}$$

For arbitrary $r\in\mathcal{R}$, the *control strategy* is given by

$$u(t)=-\gamma_\rho\big(k(t),Cx(t)-r(t),\xi(t)\big),\quad k(t)=\alpha\big(\varphi^2(t)\|Cx(t)-r(t)\|^2\big). \tag{4.6}$$

REMARK 4.1.
(i) If $s(\Gamma)$ is known *a priori*, then the function $\nu\colon k\mapsto -s(\Gamma)k$ is sufficient. If $s(\Gamma)$ is unknown, then the function $\nu\colon k\mapsto k\cos k$ suffices. In the latter case, the rôle of the function ν is similar to that of a "Nussbaum" function in adaptive control. Note, however, that the requisite property (4.2) is less restrictive than (a) the "Nussbaum property"

$$\limsup_{k\to\infty}\frac{1}{k}\int_0^k\nu(\kappa)\,\mathrm{d}\kappa=\infty,\qquad \liminf_{k\to\infty}\frac{1}{k}\int_0^k\nu(\kappa)\,\mathrm{d}\kappa=-\infty,$$

as required in [17], for example, or (b) the stronger "scaling invariant Nussbaum property", as required in [9], for example.
(ii) In the specific case of a system of relative degree $\rho=2$, writing $e(t)=Cx(t)-r(t)$ and omitting the argument t for simplicity, the control strategy takes the explicit form

$$\left.\begin{aligned}&u=\mu\,\nu(k)e-\mu\big[(\nu'(k)\|e\|)^2+(\nu(k))^2\big]\,\big(\alpha'(\varphi^2\|e\|^2)\big)^2\,[1+\|\xi\|^2]\theta\\ &k=\alpha\big(\varphi^2\|e\|^2\big),\qquad \theta=\xi-\nu(k)e\,,\\ &\dot\xi=-\mu\,\xi+u,\qquad \xi(0)=\xi^0.\end{aligned}\right\} \tag{4.7}$$

We will adopt this controller in the example in Section 6.

4.3. Well-posedness of the closed-loop system. The conjunction of the filter (4.1) and the feedback (4.6) applied to (3.1) yields the closed-loop initial-value problem

$$\left.\begin{aligned} &\dot{x}(t) = A\,x(t) + f(p(t),(TCx)(t),x(t)) - B\gamma_\rho(k(t),Cx(t)-r(t),\xi(t)),\\ &\dot{\xi}(t) = F\xi(t) - G\gamma_\rho(k(t),Cx(t)-r(t),\xi(t)),\\ &k(t) = \alpha\big(\varphi^2(t)\|Cx(t)-r(t)\|^2\big)\,,\\ &x|_{[-h,0]} = x^0 \in C([-h,0],\mathbb{R}^{\rho m}), \quad \xi(s)=\xi^0\in\mathbb{R}^{(\rho-1)m} \quad \forall s\in[-h,0]\,. \end{aligned}\right\} \tag{4.8}$$

By a solution of (4.8) on $[-h,\omega)$ we mean a function $(x,\xi)\in C([-h,\omega),\mathbb{R}^{\rho m}\times\mathbb{R}^{(\rho-1)m}$, with $0<\omega\le\infty$, $x|_{[-h,0]}=x^0$ and $\xi(s)=\xi^0$ for all $s\in[-h,0]$, such that $(x,\xi)|_{[0,\omega)}$ is absolutely continuous, satisfies the differential equations in (4.8) for almost all $t\in[0,\omega)$ and avoids the singularity in α in the sense that $\varphi(t)\|Cx(t)-r(t)\|<1$ for all $t\in[0,\omega)$. To answer affirmatively the question of well-posedness of the closed-loop, we provide an existence theorem for a class of initial-value problems of sufficient generality to incorporate (4.8). For $h\ge 0$, consider the general initial-value problem

$$\left.\begin{aligned} \dot{\zeta}(t) &= Z(t,(\widehat{T}\zeta)(t),\zeta(t)), && \zeta(t)\in\mathcal{D},\\ \zeta|_{[-h,0]} &= \zeta^0\in C([-h,0],\mathbb{R}^N), && \zeta^0(0)\in\mathcal{D}, \end{aligned}\right\} \tag{4.9}$$

where $\mathcal{D}\subset\mathbb{R}^N$ is a non-empty, open set, $Z\colon[-h,\infty)\times\mathbb{R}^q\times\mathcal{D}\to\mathbb{R}^N$ is a Carathéodory function and $\widehat{T}$ is a causal operator of class $\mathcal{T}_h$. By a solution of (4.9) on $[-h,\omega)$ we mean a function $\zeta\in C([-h,\omega),\mathbb{R}^N)$, with $0<\omega\le\infty$, and $\zeta|_{[-h,0]}=\zeta^0$ such that $\zeta|_{[0,\omega)}$ is absolutely continuous and satisfies the differential equations in (4.9) for almost all $t\in[0,\omega)$ and $\zeta(t)\in\mathcal{D}$ for all $t\in[0,\omega)$. A solution of (4.8) or of (4.9) is *maximal* if, and only if, it has no proper right extension that is also a solution.

THEOREM 4.2. *Let $\mathcal{D}\subset\mathbb{R}^N$ be non-empty and open, let $\widehat{T}$ be an operator of class $\mathcal{T}$ and let $Z\colon[-h,\infty)\times\mathbb{R}^q\times\mathcal{D}\to\mathbb{R}^N$ be a Carathéodory function. Then, for each $\zeta^0\in C([-h,0],\mathbb{R}^N)$ with $\zeta(0)\in\mathcal{D}$, there exists a solution $\zeta\colon[-h,\omega)\to\mathbb{R}^N$, $\zeta([0,\omega))\subset\mathcal{D}$, of the initial-value problem* (4.9) *and every solution can be extended to a maximal solution. Moreover, if Z is locally essentially bounded and $\zeta\colon[-h,\omega)\to\mathbb{R}^N$, $\zeta([0,\omega))\subset\mathcal{D}$, is a maximal solution with $\omega<\infty$, then, for every compact set $\mathcal{K}\subset\mathcal{D}$, there exists $\hat{t}\in[0,\omega)$ such that $\zeta(\hat{t})\notin\mathcal{K}$.*

Proof. The proof is a straightforward modification of that of [5, Theorem 5]. ☐
We apply this result to our closed-loop system (4.8).

COROLLARY 4.3. *Let $(A,B,C,f,p,T,h)\in\Sigma_\rho$ with $\rho\ge 1$ and let $\varphi\in\Phi$. For every $r\in\mathcal{R}$ and $(x^0,\xi^0)\in C([-h,0],\mathbb{R}^{\rho m}\times\mathbb{R}^{(\rho-1)m})$, application of the feedback* (4.6) *in conjunction with the filter* (4.1) *to the system* (3.1) *yields the initial-value problem* (4.8) *which has a solution and every solution can be extended to a maximal solution. If a maximal solution of* (4.8) *on $[-h,\omega)$ is bounded and such that the associated gain function k is also bounded, then $\omega=\infty$.*

The proof is in the Appendix.

5. Main Results.

5.1. Preliminary lemmas. Let $(A,B,C,f,p,T,h)\in\Sigma_\rho$ with $\rho\ge 2$. Rewriting the conjunction of the nonlinear system (3.1) and the filter (4.1) as

$$\left.\begin{aligned} \begin{bmatrix}\dot{x}(t)\\ \dot{\xi}(t)\end{bmatrix} &= \begin{bmatrix}A & 0\\ 0 & F\end{bmatrix}\begin{bmatrix}x(t)\\ \xi(t)\end{bmatrix} + \begin{bmatrix}I\\ 0\end{bmatrix} f(p(t),(Ty)(t),x(t)) + \begin{bmatrix}B\\ G\end{bmatrix}u(t),\\ y(t) &= [\,C\,\vdots\,0\,]\begin{bmatrix}x(t)\\ \xi(t)\end{bmatrix}, \end{aligned}\right\} \tag{5.1}$$

we have the following technicality, a proof of which can be found in the Appendix.

LEMMA 5.1. *For system* (5.1), *there exist* $K \in \mathbb{R}^{\rho m\times(\rho-1)m}$ *and* $N \in \mathbb{R}^{(\rho-1)m\times\rho m}$ *such that*

$$L := \begin{bmatrix} C & 0 \\ N & -NK \\ 0 & I \end{bmatrix} \in \mathbb{R}^{(2\rho-1)m\times(2\rho-1)m}$$

is invertible and

$$L \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} L^{-1} = \begin{bmatrix} A_1 & A_2 & \tilde\Gamma \\ A_3 & A_4 & 0 \\ 0 & 0 & F \end{bmatrix}, \quad L \begin{bmatrix} B \\ G \end{bmatrix} = \begin{bmatrix} 0 \\ G \end{bmatrix}, \quad [\,C \,\vdots\, 0\,]\, L^{-1} = [\,I \,\vdots\, 0 \,\vdots\, 0\,],$$

where $\tilde\Gamma := \left[\Gamma \,\vdots\, 0\right] \in \mathbb{R}^{m\times(\rho-1)m}$, $\Gamma := CA^{\rho-1}B$ *and* $A_4 \in \mathbb{R}^{(\rho-1)m\times(\rho-1)m}$ *is such that* $\operatorname{spec}(A_4) \subset \mathbb{C}_-$.

In view of Lemma 5.1, there exist K and N such that, under the coordinate change

$$\begin{bmatrix} y(t) \\ z(t) \\ \xi(t) \end{bmatrix} = L \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}, \quad \begin{bmatrix} y^0 \\ z^0 \\ \xi^0 \end{bmatrix} = L \begin{bmatrix} x^0 \\ \xi^0 \end{bmatrix}, \quad L := \begin{bmatrix} C & 0 \\ N & -NK \\ 0 & I \end{bmatrix}, \tag{5.2}$$

the conjunction (5.1) of system (3.1) and filter (4.1) can be represented by

$$\left.\begin{aligned} &\dot y(t) = A_1 y(t) + A_2 z(t) + Cf(p(t),(Ty)(t),x(t)) + \Gamma\xi_1(t), \\ &\dot z(t) = A_3 y(t) + A_4 z(t) + Nf(p(t),(Ty)(t),x(t)), \\ &\dot\xi(t) = F\xi(t) + Gu(t), \\ &(y,z,\xi)|_{[-h,0]} = (y^0,z^0,\xi^0) \in C([-h,0],\mathbb{R}^m\times\mathbb{R}^{(\rho-1)m}\times\mathbb{R}^{(\rho-1)m}), \end{aligned}\right\} \tag{5.3}$$

where $A_4 \in \mathbb{R}^{(\rho-1)m\times(\rho-1)m}$ has spectrum in $\mathbb{C}_-$. If $(x,\xi)\colon [0,\omega) \to \mathbb{R}^{\rho m}\times\mathbb{R}^{(\rho-1)m}$ is a maximal solution of the nonlinearly-perturbed closed-loop system (4.8), then, in view of (5.3) and writing

$$y(t) = Cx(t), \qquad e(t) = y(t) - r(t), \quad e|_{[-h,0]} = e^0 = y^0 - r|_{[-h,0]}\,, \tag{5.4}$$

we arrive at the following equivalent to (4.8)

$$\left.\begin{aligned} &\dot e(t) = A_1 e(t) + A_2 z(t) + f_1(t) + \Gamma\xi_1(t), \\ &\dot z(t) = A_3 e(t) + A_4 z(t) + f_2(t), \\ &\dot\xi(t) = F\xi(t) - G\gamma_\rho(k(t),e(t),\xi(t)), \\ &k(t) = \alpha\big(\varphi^2(t)\|e(t)\|^2\big), \\ &(e,z,\xi)|_{[-h,0]} = (e^0,z^0,\xi^0) \in C([-h,0],\mathbb{R}^m\times\mathbb{R}^{(\rho-1)m}\times\mathbb{R}^{(\rho-1)m}), \end{aligned}\right\} \tag{5.5}$$

where the functions f_1 and f_2 are given by

$$\left.\begin{aligned} &f_1(t) := A_1 r(t) + Cf(p(t),(Ty)(t),x(t)) - \dot r(t), \\ &f_2(t) := A_3 r(t) + Nf(p(t),(Ty)(t),x(t)). \end{aligned}\right\} \tag{5.6}$$

Since $(\varphi(t)\|e(t)\|)^2 < 1$, the properties of $\varphi \in \Phi$ yield boundedness of the function e which, together with boundedness of r, implies boundedness of y. Since T is of class $\mathcal{T}_h$ and y is bounded, Ty is essentially bounded. By boundedness of r, essential boundedness of $\dot r$ and p, and Assumption (A4), we may now conclude (essential) boundedness of the functions f_1 and f_2. Observing that A_4 is Hurwitz and f_2 bounded, the second of the differential equations in (5.5) yields boundedness of z. These observations are recorded in the following lemma.

LEMMA 5.2. *Let* $(A,B,C,f,p,T,h) \in \Sigma_\rho$ *with* $\rho \geq 2$. *Let* $\varphi \in \Phi$, $r \in \mathcal{R}$ *and* $(x^0,\xi^0) \in C([-h,0],\mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m})$. *If* $(x,\xi)\colon [-h,\omega) \to \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m}$ *is a maximal solution of* (4.8), *then the functions* y, z *and* e, *given by* (5.2) *and* (5.4), *are bounded. Furthermore, the functions* f_1 *and* f_2, *given by* (5.6), *are essentially bounded and bounded, respectively.*

The proofs of our main results (Theorems 5.4 and 5.5 below) rely crucially on a further technicality: the signals $\theta_i = \mu^{1-i}\xi_i + \gamma_i(k,e,\pi_{i-1}\xi)$, $i = 1,...,\rho-1$, are bounded. More precisely, we have the following (with proof in the Appendix).

LEMMA 5.3. *Let* $(A,B,C,f,p,T,h) \in \Sigma_\rho$ *with* $\rho \geq 2$. *Let* $\varphi \in \Phi$, $r \in \mathcal{R}$ *and* $(x^0,\xi^0) \in C([-h,0],\mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m})$. *If* $(x,\xi)\colon [-h,\omega) \to \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m}$ *is a maximal solution of* (4.8), *then the function* $\theta = (\theta_1,\ldots,\theta_{\rho-1})\colon [0,\omega) \to \mathbb{R}^{(\rho-1)m}$ *is bounded, where*

$$\theta_i(t) := \mu^{1-i}\,\xi_i(t) + \gamma_i(k(t),e(t),\pi_{i-1}\xi(t))\,, \qquad i = 1,\ldots,\rho-1, \tag{5.7}$$

with the notational convention $\gamma_1(k,e,\pi_0\xi) := \gamma_1(k,e)$.

5.2. Relative degree 1 case. We are now in a position to state our main result for the case when the system has relative degree 1; in this case, a filter is not necessary and the controller (4.6) simplifies to

$$u(t) = \nu(k(t))(Cx(t) - r(t)), \quad k(t) = \alpha\big(\varphi^2(t)\|Cx(t) - r(t)\|^2\big)\,. \tag{5.8}$$

The closed-loop initial-value problem then becomes

$$\left.\begin{aligned} &\dot{x}(t) = Ax(t) + B\nu(k(t))(Cx(t) - r(t)) + f(p(t),T(Cx)(t),x(t)),\\ &k(t) = \alpha\big(\varphi^2(t)\|Cx(t) - r(t)\|^2\big)\,,\\ &x|_{[-h,0]} = x^0 \in C([-h,0],\mathbb{R}^m). \end{aligned}\right\} \tag{5.9}$$

THEOREM 5.4. *Let* $(A,B,C,f,p,T,h) \in \Sigma_1$ *and* $\varphi \in \Phi$ *with associated performance funnel* $\mathcal{F}_\varphi$. *For each reference signal* $r \in \mathcal{R}$, *and initial data* $(x^0,\xi^0) \in C([-h,0],\mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m})$, *application of the feedback* (5.8) *to* (3.1) *yields the initial-value problem* (5.9) *which has a solution and every solution can be maximally extended. Every maximal solution* $x\colon [-h,\omega) \to \mathbb{R}^m$ *has the properties:*
(i) $\omega = \infty$;
(ii) x, k *and* u *are bounded;*
(iii) *the tracking error evolves within the funnel* $\mathcal{F}_\varphi$ *and is bounded away from the funnel boundary, i.e. there exists* $\varepsilon > 0$ *such that, for all* $t \geq 0$, $\varphi(t)\|Cx(t)-r(t)\| \leq 1-\varepsilon$.

The proof of Theorem 5.4 follows easily by modifying (all vestiges of the filter equations are excised) the proof of Theorem 5.5 below. The latter proof is in the Appendix.

5.3. Relative degree $\rho \geq 2$ case. We now arrive at the main result of the paper (with proof in the Appendix).

THEOREM 5.5. *Let* $(A,B,C,f,p,T,h) \in \Sigma_\rho$ *with* $\rho \geq 2$ *and let* $\varphi \in \Phi$ *with associated performance funnel* $\mathcal{F}_\varphi$. *For each reference signal* $r \in \mathcal{R}$ *and initial data* $(x^0,\xi^0) \in C([-h,0],\mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m})$, *application of the feedback* (4.6), *in conjunction with the filter* (4.1), *to* (3.1) *yields the initial-value problem* (4.8) *which has a solution and every solution can be maximally extended. Every maximal solution* $(x,\xi)\colon [-h,\omega) \to \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m}$ *has the properties:*

(i) $\omega = \infty$;
(ii) x, ξ, k *and* u *are bounded;*
(iii) *the tracking error evolves within the funnel* $\mathcal{F}_\varphi$ *and is bounded away from the funnel boundary, i.e. there exists* $\varepsilon > 0$ *such that, for all* $t \geq 0$, $\varphi(t)\|Cx(t)-r(t)\| \leq 1-\varepsilon$.

6. Example. We illustrate the controller strategy (4.6) applied to the following single-input, single-output system of relative degree $\rho = 2$:

$$\ddot{y}(t) + b_0 \sin y(t) + b_1 y(t)|y(t)| + (T_{a,b}\, y)(t) = b_2\, u(t)\,, \tag{6.1}$$

where b_0, b_1 and $b_2 \neq 0$ are unknown real parameters and $T_{a,b}$ represents the backlash operator as defined in Section 3.1.5, with parameters $a > 0$ and $b \in [-a, a]$. Equation (6.1) is equivalent to (3.1) with

$$x(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ b_2 \end{bmatrix}, \quad C = [1 \vdots 0], \quad f(p, w, x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} w,$$

and the operator T given by $(Ty)(t) = b_0 \sin y(t) + b_1 y(t)|y(t)| + (T_{a,b} y)(t)$, $t \in \mathbb{R}_+$. Setting $h = 0$ and $p = 0$, the resulting system $(A, B, C, f, 0, T, 0)$ is of class Σ_2.

Fix $\tau > 0$ arbitrarily and define $\varphi \in \Phi$ by

$$t \mapsto \varphi(t) = \begin{cases} 20(1 - (\frac{t}{\tau} - 1)^2), & 0 \leq t < \tau \\ 20, & t \geq \tau. \end{cases} \tag{6.2}$$

Evolution within the associated performance funnel $\mathcal{F}_\varphi$ ensures a tracking accuracy $|e(t)| < 0.05$ for all $t \geq \tau$. Choosing $\nu\colon k \mapsto k \cos k$, $\alpha\colon s \mapsto (1 - s)^{-1}$, writing $e(t) = y(t) - r(t)$ and suppressing the argument t for simplicity, the control strategy (4.7), with $\xi^0 = 0$, becomes

$$\left.\begin{array}{ll} u = \mu(k\cos k)e - \mu\left[(\cos k - k \sin k)^2\, e^2 + k^2 \cos^2 k\right] k^4\, [1 + \xi^2]\theta & \\ k = \left[1 - \varphi^2 e^2\right]^{-1}, & \theta = \xi - (k \cos k)e \\ \dot{\xi} = -\mu\, \xi + u, & \xi(0) = 0. \end{array}\right\} \tag{6.3}$$

For purposes of illustration, as reference signal $r \in \mathcal{R}$, we take the first component ζ_1 of the solution (chaotic and bounded, see [15, Appendix C]) of the following Lorenz system of equations:

$$\left.\begin{array}{ll} \dot{\zeta}_1(t) = \frac{1}{2}\zeta_2(t) - \zeta_1(t), & \zeta_1(0) = \frac{1}{2}, \\ \dot{\zeta}_2(t) = \frac{28}{5}\zeta_1(t) - \frac{1}{10}\zeta_2(t) - 2\zeta_1(t)\zeta_3(t), & \zeta_2(0) = 0, \\ \dot{\zeta}_3(t) = 2\zeta_1(t)\zeta_2(t) - \frac{8}{30}\zeta_3(t), & \zeta_3(0) = 3. \end{array}\right\} \tag{6.4}$$

Setting $b_0 = \frac{1}{2}$, $b_1 = 1 = b_2$, $\mu = 10$, $\tau = 50$ and adopting backlash hysteresis with parameters $a = 1/2$, $b = 0$ and initial data $(y(0), \dot{y}(0)) = (0, 0)$, the behaviour of the closed-loop system (6.1)–(6.3) is depicted in Figure 5.

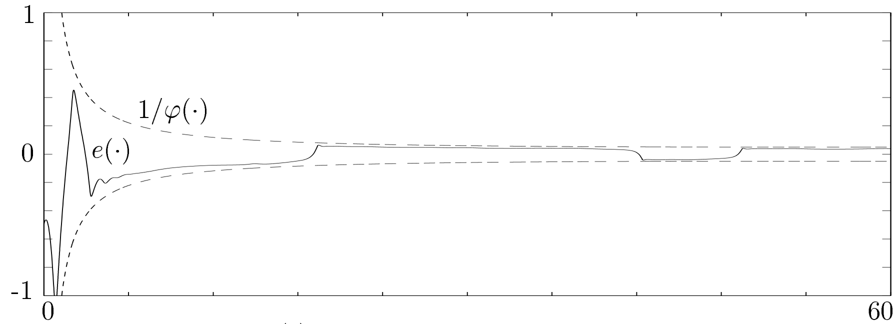
7. Appendix.

7.1. Proof of Lemma 3.5.

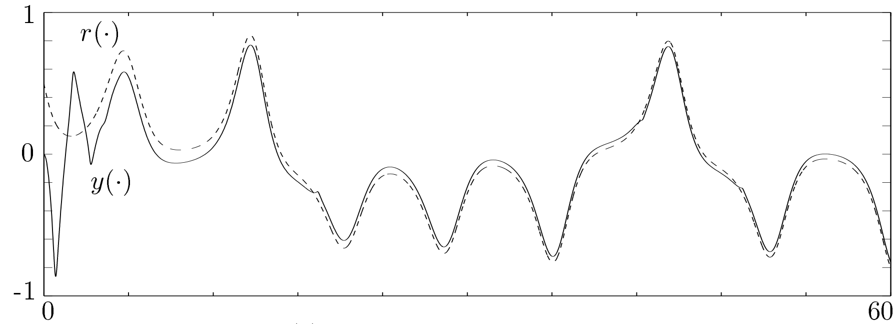
Parts of the following proof are implicit in the proofs of [7, Lemma 4.1.1] and [8, Propositions 11.5.1 and 11.5.2] (in a general context of nonlinear systems); here, we provide a simple, self-contained proof in the restricted context of linear systems.

Step (i): First note that

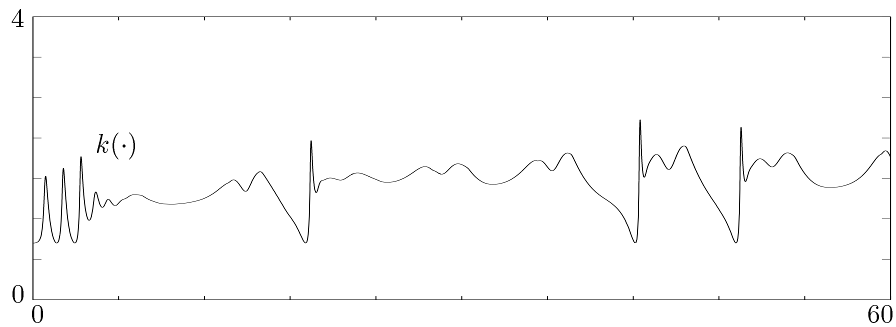
$$\mathcal{CB} = \begin{bmatrix} 0 & & \Gamma \\ & \iddots & \\ \Gamma & & * \end{bmatrix},$$



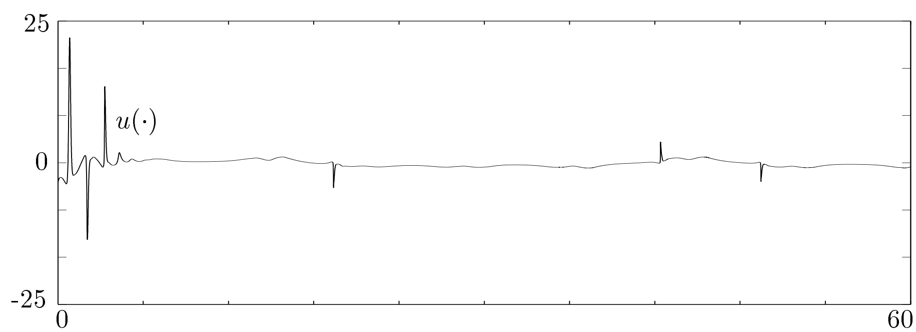
(a) The funnel and tracking error e



(b) The reference r and output y



(c) The function k



(d) The control u

FIG. 6.1. *Tracking of a Lorenz component reference signal; system* (6.1) *with unknown sign* $b_2 \neq 0$ *and control strategy* (6.3).

and, since Γ is invertible, we see that $\mathcal{C}\mathcal{B} \in \mathrm{GL}_{\rho m}(\mathbb{R})$. Furthermore, $\mathcal{N}\mathcal{B} = 0$. Assertion (i) then follows from the observation that, for any $x \in \mathbb{R}^n$, we have $v := (I - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C})x \in \ker \mathcal{C}$ and $w := \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}x \in \mathrm{im}\, \mathcal{B}$, and so $x = v + w$.

Step (ii): We now prove Assertion (ii). It is clear that $\mathcal{U}^{-1} = [\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \vdots \mathcal{V}]$. It is also immediate that $\hat{B} := \mathcal{U}\tilde{B}$ and $\hat{C} := \tilde{C}\mathcal{U}^{-1}$ have the structure given in (3.8). Furthermore ,

$$\mathcal{U}\tilde{A} = \hat{A}\mathcal{U} \tag{7.1}$$

for some $\hat{A}$ of the form:

$$\hat{A} = \begin{bmatrix} 0 & I & 0 & \dots & 0 & 0 \\ 0 & 0 & I & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \dots & 0 & I & 0 \\ R_1 & R_2 & \dots & R_{\rho-1} & R_\rho & S \\ P_1 & P_2 & \dots & P_{\rho-1} & P_\rho & Q \end{bmatrix},$$

with $R_i \in \mathbb{R}^{m\times m}$, $P_i \in \mathbb{R}^{(n-\rho m)\times m}$, $i = 1, ..., \rho$, $S \in \mathbb{R}^{m\times(n-\rho m)}$ and $Q = \mathcal{N}\tilde{A}\mathcal{V} \in \mathbb{R}^{(n-\rho m)\times(n-\rho m)}$. If $\rho = 1$, then $\hat{A}$ takes the form shown in (3.9).

Recalling that $\mathcal{N}\mathcal{B} = 0$, we see that

$$[P_1 \vdots \cdots \vdots P_\rho] = \mathcal{N}\tilde{A}\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} = [0 \vdots \cdots \vdots 0 \vdots \mathcal{N}\tilde{A}^\rho \tilde{B}] \begin{bmatrix} * & & \Gamma^{-1} \\ & \cdot^{\cdot^{\cdot}} & \\ \Gamma^{-1} & & 0 \end{bmatrix},$$

hence $P_i = 0$ for $i = 2, \dots, \rho$. Writing $P = P_1$, it follows that $\hat{A}$ takes the form in (3.8) and $P = \mathcal{N}\tilde{A}^\rho \tilde{B}\Gamma^{-1}$.

Step (iii): Finally we prove part (iii) of the lemma. Writing

$$M_1(s) = \begin{bmatrix} sI - \tilde{A} & \tilde{B} \\ \tilde{C} & 0 \end{bmatrix}, \quad M_2(s) = \begin{bmatrix} \mathcal{U} & 0 \\ 0 & I \end{bmatrix} M_1(s) \begin{bmatrix} \mathcal{U}^{-1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} sI - \hat{A} & \hat{B} \\ \hat{C} & 0 \end{bmatrix},$$

and

$$M_3(s) = \begin{bmatrix} \hat{C} & 0 \\ \hat{A} - sI & -\hat{B} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & \dots & 0 & 0 & 0 \\ -sI & I & 0 & \dots & 0 & 0 & 0 \\ 0 & -sI & I & & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & \dots & -sI & I & 0 & 0 \\ R_1 & R_2 & \dots & R_{\rho-1} & R_\rho - sI & S & -\Gamma \\ P & 0 & \dots & 0 & 0 & Q - sI & 0 \end{bmatrix},$$

we see that $|\det M_1(s)| = |\det M_2(s)| = |\det M_3(s)| = |\det \Gamma \, \det(sI - Q)|$.

By the minimum-phase property of $(\tilde{A}, \tilde{B}, \tilde{C})$, we have $\det(M_1(s)) \neq 0$ for all $s \in \mathbb{C} \setminus \mathbb{C}_-$ and so $\det(sI - Q) \neq 0$ for all $s \in \mathbb{C} \setminus \mathbb{C}_-$. It follows that $\mathrm{spec}(Q) \subset \mathbb{C}_-$ and hence Assertion (iii) holds. □

7.2. Proof of Corollary 4.3. Introducing the open set

$$\mathcal{D} := \left\{ (x,\xi,\eta) \in \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m} \times \mathbb{R} \;\middle|\; \big(\varphi(|\eta|)\,\|Cx - r(|\eta|)\|\big)^2 < 1 \right\},$$

and defining, on $\mathcal{D}$,

$$\gamma_\rho^* \colon (x,\xi,\eta) \mapsto \gamma_\rho\left(1/(1-(\varphi(|\eta|)\,\|Cx - r(|\eta|)\|)^2)\,,\; Cx - r(|\eta|), \xi\right),$$

the initial-value problem (4.8) may be recast on $\mathcal{D}$ as

$$\left.\begin{aligned} &\dot{x}(t) = Ax(t) + f(p(t), T(Cx)(t), x(t)) - B\gamma_\rho^*(x(t),\xi(t),\eta(t)),\\ &\dot{\xi}(t) = F\xi(t) - G\gamma_\rho^*(x(t),\xi(t),\eta(t)),\\ &\dot{\eta}(t) = 1,\\ &(x,\xi,\eta)|_{[-h,0]} = (x^0,\xi^0,0) \in C([-h,0], \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m} \times \mathbb{R}). \end{aligned}\right\} \tag{7.2}$$

Setting $\zeta = (x,\xi,\eta)$ and defining the Carathéodory function

$$Z \colon [-h,\infty) \times \mathbb{R}^q \times \mathbb{R}^{2(\rho-1)m+1} \to \mathbb{R}^{(2\rho-1)m+1}$$

$$(t,w,\zeta) \mapsto Z(t,w,\zeta) := \begin{bmatrix} A & 0 & 0\\ 0 & F & 0\\ 0 & 0 & 0 \end{bmatrix} \zeta + \begin{bmatrix} I\\ 0\\ 0 \end{bmatrix} f(p(t),w,x) - \begin{bmatrix} B\\ G\\ 0 \end{bmatrix} \gamma_\rho^*(\zeta) + \begin{bmatrix} 0\\ 0\\ 1 \end{bmatrix}$$

we can rewrite (7.2) as follows

$$\dot{\zeta}(t) = Z(t, (\widehat{T}\zeta)(t), \zeta(t)) \qquad \zeta|_{[-h,0]} = \zeta^0 \in C([-h,0], \mathbb{R}^{(2\rho-1)m+1}), \tag{7.3}$$

where the operator $\widehat{T}$, given by $(\widehat{T}\zeta)(t) = (TCx)(t)$, is of class $\mathcal{T}_h$. We then apply the existence result, Theorem 4.2, to conclude: (i) the existence of a solution $t \mapsto \zeta(t) \in \mathcal{D}$ to (7.2) and (ii) every solution can be extended to a maximal solution $\zeta \colon [-h,\omega) \to \mathcal{D}$. Furthermore, if there exists a compact set $\mathcal{C} \subset \mathcal{D}$ such that $(x(t),\xi(t),\eta(t)) \in \mathcal{C}$ for all $t \in [0,\omega)$, then $\omega = \infty$.

Clearly, if $\zeta = (x,\xi,\eta) \colon [-h,\omega) \to \mathcal{D}$ is a solution of (7.3), then $(x,\xi) \colon [-h,\omega) \to \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m}$ is a solution of (4.8); conversely, if $(x,\xi) \colon [-h,\omega) \to \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m}$ is a solution of (4.8), then $\zeta = (x,\xi,\eta) \colon [-h,\omega) \to \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m} \times \mathbb{R}$, with component η given by $\eta(t) = t$, is a solution of (7.3). We may now conclude that, for each $(x^0,\xi^0) \in C([-h,0], \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m})$, (4.8) has a solution and every solution can be maximally extended.

Let $(x,\xi) \colon [-h,\omega) \to \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m}$ be a maximal solution of (4.8) (and so $t \mapsto \zeta(t) = (x(t),\xi(t),t)$ is a maximal solution of (4.9)). Assume that (x,ξ) is bounded and that the gain function $t \mapsto k(t) = \alpha\big(\varphi^2(t)\|Cx(t) - r(t)\|^2\big)$ is also bounded. Then there exist $c > 0$ and $\varepsilon > 0$ such that $\|(x(t),\xi(t))\| \le c$ and $\varphi(t)\|Cx(t) - r(t)\| \le 1 - \varepsilon$ for all $t \in [0,\omega)$. Seeking a contradiction, suppose that $\omega < \infty$. It then follows that $\mathcal{K} := \big\{(x,\xi,\eta) \in \mathcal{D} \,\big|\, \varphi(|\eta|)\|Cx - r(|\eta|)\| \le 1 - \varepsilon,\ \|(x,\xi)\| \le c,\ \eta \in [-h,\omega]\big\}$ is a compact subset of $\mathcal{D}$ which contains the trajectory $\zeta([-h,\omega))$ of the maximal solution ζ of (4.9). This contradicts the last assertion of Theorem 4.2, and so $\omega = \infty$. This completes the proof. □

7.3. Proof of Lemma 5.1. Define

$$K := \big[[\mu I + A]^{\rho-2}B \,\vdots\, [\mu I + A]^{\rho-3}B \,\vdots\, \cdots \,\vdots\, [\mu I + A]B \,\vdots\, B\big] \in \mathbb{R}^{\rho m \times (\rho-1)m}$$

and note that

$$AK - KF = \big[[\mu I + A]^{\rho-1}B \,\vdots\, 0 \,\vdots\, \cdots \,\vdots\, 0\big], \quad KG = B \quad \text{and} \quad CK = 0.$$

Writing $\tilde{B} := (\mu I + A)^{\rho-1}B$, we have $C\tilde{B} = CA^{\rho-1}B = \Gamma$ and so the triple $(A, \tilde{B}, C)$ defines a linear system of relative degree one. Let $V \in \mathbb{R}^{\rho m \times (\rho-1)m}$ be such that $\operatorname{im} V = \ker C$. By Lemma 3.5 applied in the context of the system $(A, \tilde{B}, C)$, the matrix $\begin{bmatrix} C \\ N \end{bmatrix}$, with $N := (V^T V)^{-1} V^T [I - \tilde{B}\Gamma^{-1}C]$, is invertible, with inverse $\big[\tilde{B}\Gamma^{-1} \,\vdots\, V\big]$. Writing

$$L = \begin{bmatrix} C & 0 \\ N & -NK \\ 0 & I \end{bmatrix} \quad \text{with} \quad L^{-1} = \begin{bmatrix} \tilde{B}\Gamma^{-1} & V & K \\ 0 & 0 & I \end{bmatrix}$$

and recalling that $KG = B$, $CB = 0$ and $CK = 0$, we have

$$L \begin{bmatrix} B \\ G \end{bmatrix} = \begin{bmatrix} 0 \\ G \end{bmatrix} \quad \text{and} \quad \big[C \,\vdots\, 0\big] L^{-1} = \big[I \,\vdots\, 0\big].$$

Moreover, noting that $CAK = \big[\Gamma \,\vdots\, 0 \,\vdots\, \cdots \,\vdots\, 0\,\big] =: \tilde{\Gamma}$ and $N[AK - KF] = 0$, we have

$$L \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} L^{-1} = \begin{bmatrix} CA\tilde{B}\Gamma^{-1} & CAV & CAK \\ NA\tilde{B}\Gamma^{-1} & NAV & N[AK - KF] \\ 0 & 0 & F \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & \tilde{\Gamma} \\ A_3 & A_4 & 0 \\ 0 & 0 & F \end{bmatrix},$$

where $\tilde{\Gamma} = [\Gamma \,\vdots\, 0 \,\vdots\, \cdots \,\vdots\, 0]$. It remains to show that A_4 has spectrum in $\mathbb{C}_-$. Writing

$$M_4(s) = \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} \quad \text{and} \quad M_5(s) = \begin{bmatrix} sI - A & 0 & B \\ 0 & sI - F & -G \\ C & 0 & 0 \end{bmatrix},$$

we have

$$M_6(s) := \begin{bmatrix} I & K & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} M_5(s) \begin{bmatrix} I & K & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} sI - A & AK - KF & 0 \\ 0 & sI - F & -G \\ C & 0 & 0 \end{bmatrix}.$$

In view of the particular structure of F, G and $AK - KF$, it is readily verified that $|\det M_6(s)| = |\det M_7(s)|$, where $M_7(s) = \begin{bmatrix} sI - A & [\mu I + A]^{\rho-1}B \\ C & 0 \end{bmatrix}$. Define

$$M_8(s) := \begin{bmatrix} C & 0 \\ N & 0 \\ 0 & I \end{bmatrix} M_7(s) \begin{bmatrix} \tilde{B}\Gamma^{-1} & V & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} sI - A_1 & -A_2 & \Gamma \\ -A_3 & sI - A_4 & 0 \\ I & 0 & 0 \end{bmatrix}.$$

By the minimum-phase property of the triple (A, B, C) (recall Remark 3.4(ii)), for all $s \in \mathbb{C} \setminus \mathbb{C}_-$, we have $\det M_4(s) \neq 0$. We may now conclude that, for all $s \in \mathbb{C} \setminus \mathbb{C}_-$,

$$\begin{aligned} |\det \Gamma \, \det(sI - A_4)| &= |\det M_8(s)| = |\det M_7(s)| \\ &= |\det M_6(s)| = |\det M_5(s)| = |\det(sI - F) \, \det M_4(s)| \neq 0, \end{aligned}$$

and so $\operatorname{spec}(A_4) \subset \mathbb{C}_-$. This completes the proof. □

7.4. Proof of Lemma 5.3. Assume that $(x,\xi)\colon [-h,\omega) \to \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m}$ is a maximal solution of (4.8). Write $y(t) = Cx(t)$ and $e(t) = y(t) - r(t)$ for all $t \in [-h,\omega)$. By Lemma 5.1, there exists an invertible linear transformation L under which the closed-loop system (4.8) may be expressed in the form (5.5), wherein, by Lemma 5.2, e and z are bounded and the functions f_1 and f_2 given by (5.6) are essentially bounded and bounded respectively. By boundedness of z, essential boundedness of f_1 and the first of equations (5.5), we may infer the existence of $c_1 > 0$ such that

$$\|\dot e(t)\| \le c_1\big(1 + \|\xi_1(t)\|\big) \quad \text{for a.a. } t \in [0,\omega).$$

By boundedness of φ, e and essential boundedness of $\dot\varphi$, there exists $c_2 > 0$ such that

$$\begin{aligned}|\dot k(t)| &= 2\alpha'(\varphi^2(t)\|e(t)\|^2)\big|\varphi^2(t)\langle e(t),\dot e(t)\rangle + \varphi(t)\dot\varphi(t)\|e(t)\|^2\big| \\ &\le c_2\,\alpha'(\varphi^2(t)\|e(t)\|^2)\,(1 + \|\xi_1(t)\|) \quad \text{for a.a. } t \in [0,\omega).\end{aligned}$$

By properties of α (see Section 4.2), there exists $\delta > 0$ such that, for all $s \in [0,1)$, $\alpha'(s) \ge \delta$, and so there exists a constant $c_3 > 0$ such that

$$\|(\dot k(t),\dot e(t))\|^2 \le c_3\,\Delta(t) \quad \text{for a.a. } t \in [0,\omega),$$

where, for notational convenience, we write $\Delta(t) := \big(\alpha'(\varphi^2(t)\|e(t)\|^2)\big)^2\,\big(1 + \|\xi_1(t)\|^2\big)$. Then, invoking (4.4), (5.7), and writing $c_{4,1} := c_3/\mu > 0$, we have,

$$\begin{aligned}\langle\theta_1(t),\dot\theta_1(t)\rangle &\le \langle\theta_1(t)\,,\,-\mu\xi_1(t) + \xi_2(t)\rangle + \|\theta_1(t)\|\|D\gamma_1(k(t),e(t))\|\|(\dot k(t),\dot e(t))\| \\ &\le \langle\theta_1(t)\,,\,-\mu\theta_1(t) + \mu\gamma_1(k(t),e(t))\rangle + \langle\theta_1(t),\xi_2(t)\rangle \\ &\qquad +\sqrt{\mu}\,\|\theta_1(t)\|\,\|D\gamma_1(k(t),e(t))\|\,\sqrt{(c_3/\mu)\,\Delta(t)} \\ &\le c_{4,1} - \mu\|\theta_1(t)\|^2 + \langle\theta_1(t),\xi_2(t)\rangle + \mu\langle\theta_1(t),\gamma_1(k(t),e(t))\rangle \\ &\qquad +\mu\|\theta_1(t)\|^2\,\|D\gamma_1(k(t),e(t))\|^2\,\Delta(t) \\ &= c_{4,1} - \mu\|\theta_1(t)\|^2 + \big\langle\theta_1(t),\xi_2(t) + \mu\gamma_2(k(t),e(t),\xi_1(t))\big\rangle \\ &= c_{4,1} - \mu\|\theta_1(t)\|^2 + \mu\langle\theta_1(t),\theta_2(t)\rangle \qquad \text{for a.a. } t \in [0,\omega).\end{aligned}$$

Analogous calculations yield the existence of constants $c_{4,2},\ldots,c_{4,\rho-1} > 0$, such that

$$\langle\theta_i(t),\dot\theta_i(t)\rangle \le c_{4,i} - \mu\|\theta_i(t)\|^2 + \mu\langle\theta_i(t),\theta_{i+1}(t)\rangle \quad \text{a.a. } t \in [0,\omega),\ \ i = 2,\ldots,\rho-2$$

and, using (4.5), $\langle\theta_{\rho-1}(t),\dot\theta_{\rho-1}(t)\rangle \le c_{4,\rho-1} - \mu\|\theta_{\rho-1}(t)\|^2$ for almost all $t \in [0,\omega)$. Writing $c_4 = c_{4,1} + \cdots + c_{4,\rho-1}$, we have

$$\begin{aligned}\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\,\|\theta(t)\|^2 &\le c_4 - \mu\|\theta(t)\|^2 + \mu\langle\theta_1(t),\theta_2(t)\rangle + \cdots + \mu\langle\theta_{\rho-2}(t),\theta_{\rho-1}(t)\rangle \\ &= c_4 - \mu\langle\theta(t),P\theta(t)\rangle \qquad \text{for a.a. } t \in [0,\omega),\end{aligned}$$

where P is a positive-definite, symmetric, tridiagonal matrix with all diagonal entries equal to 1 and all sub- and superdiagonal entries equal to -1/2. By positivity of P, it follows that θ is bounded. This completes the proof of the lemma. □

7.5. Proof of Theorem 5.5. Let (x^0,ξ^0) be arbitrary. By Corollary 4.3, (4.8) has a solution and every solution can be maximally extended. Let (x,ξ) be a maximal solution of (4.8) with interval of existence $[-h,\omega)$. Writing $y(t) = Cx(t)$, $e(t) = y(t) - r(t)$ for all $t \in [0,\omega)$ and invoking Lemma 5.1, there exists an invertible linear transformation L which takes (4.8) into the equivalent form (5.5)–(5.6). Introducing $\theta_1\colon [0,\omega) \to \mathbb{R}^m$ given by (5.7), viz. $\theta_1(t) = \xi_1(t) - \nu(k(t))e(t)$, then, by the first of equations (5.5), we have

$$\dot e(t) = f_3(t) + \nu(k(t))\,\Gamma\,e(t) \quad \text{for a.a. } t \in [0,\omega), \tag{7.4}$$

with $f_3(t) := A_1 e(t) + A_2 z(t) + \Gamma\theta_1(t) + f_1(t)$. By Lemmas 5.2 and 5.3, the functions y, z, e and $\theta = (\theta_1, ..., \theta_{\rho-1})$, given by (5.7), are bounded which, together with essential boundedness of f_1, implies essential boundedness of f_3. Therefore, there exists $c_5 > 0$ such that

$$\langle e(t), \dot{e}(t)\rangle \le c_5 + \nu(k(t))\, \langle e(t), \Gamma e(t)\rangle \quad \text{for a.a. } t \in [0, \omega). \tag{7.5}$$

We are now in a position to prove boundedness of k. Recalling that Γ is either positive definite or negative definite, there exist constants $\beta_0, \beta_1 > 0$ such that

$$\beta_0 \|e\|^2 \le |\langle e, \Gamma e\rangle| \le \beta_1 \|e\|^2 \quad \forall\, e \in \mathbb{R}^m.$$

Define the continuous function $\tilde{\nu}\colon \mathbb{R} \to \mathbb{R}$ as follows

$$\tilde{\nu}(k) := \begin{cases} -\beta_1 \nu(k), & s(\Gamma)\nu(k) \ge 0\,, \\ -\beta_0 \nu(k), & s(\Gamma)\nu(k) < 0\,. \end{cases}$$

Observe that

$$\nu(k)\langle e, \Gamma e\rangle \le -s(\Gamma)\tilde{\nu}(k)\|e\|^2 \quad \forall\, e \in \mathbb{R}^m\,,\ \forall\, k \ge 0\,,$$

which, together with boundedness of e, φ, essential boundedness of $\dot{\varphi}$ and (7.5), implies the existence of $c_6 > 0$ such that

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \left(\varphi(t)\|e(t)\|\right)^2 \le c_6 - 2s(\Gamma)\, \tilde{\nu}(k(t))\, \big(\varphi(t)\|e(t)\|\big)^2 \quad \text{for a.a. } t \in [0, \omega).$$

By continuity, k is bounded on $[0, \delta]$, where $0 < \delta < \min\{1, \omega\}$. Seeking a contradiction, suppose k is unbounded on $[\delta, \omega)$. By property (4.2) of ν, there exists a strictly increasing unbounded sequence (k_j) in $(\alpha(\delta), \infty)$ such that $s(\Gamma)\tilde{\nu}(k_j) \to \infty$ as $j \to \infty$. For each $j \in \mathbb{N}$, define $\tau_j := \inf\{t \in [0, \omega) |\ k(t) = k_{j+1}\}$, $\sigma_j := \sup\{t \in [0, \tau_j] |\ \tilde{\nu}(k(t)) = \tilde{\nu}(k_j)\}$ and $\tilde{\sigma}_j := \sup\{t \in [0, \tau_j] |\ k(t) = k_j\} \le \sigma_j$. Then, for all $j \in \mathbb{N}$ and all $t \in [\sigma_j, \tau_j]$, we have $k(t) \ge k_j$ and $\tilde{\nu}(k(t)) \ge \tilde{\nu}(k_j)$. Therefore,

$$(\varphi(t)\|e(t)\|)^2 \ge \alpha^{-1}(k_j) > \alpha^{-1}(\alpha(\delta)) = \delta > 0 \quad \forall\, t \in [\sigma_j, \tau_j] \ \ \forall\, j \in \mathbb{N},$$

where $\alpha^{-1}\colon [\alpha(0), \infty) \to [0, 1)$ is the inverse of the bijection $\alpha\colon [0, 1) \to \mathrm{im}(\alpha)$. Thus,

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \left(\varphi(t)\|e(t)\|\right)^2 \le c_6 - 2\delta s(\Gamma)\tilde{\nu}(k(t)) \quad \forall\, t \in [\sigma_j, \tau_j] \ \ \forall\, j \in \mathbb{N}.$$

Let $j^* \in \mathbb{N}$ be sufficiently large so that $c_6 - 2\delta s(\Gamma)\tilde{\nu}(k_{j^*}) < 0$. Then,

$$\big(\varphi(\tau_{j^*})\|e(\tau_{j^*})\|\big)^2 - \big(\varphi(\sigma_{j^*})\|e(\sigma_{j^*})\|\big)^2 < 0,$$

whence the contradiction

$$0 > \alpha\big(\varphi^2(\tau_{j^*})\|e(\tau_{j^*})\|^2\big) - \alpha\big(\varphi^2(\sigma_{j^*})\|e(\sigma_{j^*})\|^2\big) = k(\tau_{j^*}) - k(\sigma_{j^*}) \ge 0.$$

This proves boundedness of k. Since k is bounded, there exists $\varepsilon > 0$ such that $\varphi(t)\|e(t)\| \le 1 - \varepsilon$ for all $t \in [0, \omega)$. By boundedness of y, z, θ and k, it follows from the recursive construction in (5.7) that, for $i = 1, ..., \rho - 1$, γ_i and ξ_i are bounded. Consequently x and ξ are bounded and, by (4.3), (4.4) and (4.5), boundedness of γ_ρ (and hence of u) follows. Finally, by boundedness of x, ξ and k, together with Corollary 4.3, we conclude that $\omega = \infty$. This completes the proof. □

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